Behavior of Limit Cycles in Nonlinear Systems

H. Fathabadi, *Member, IACSIT*

***Abstract*—In this paper, the behavior of limit cycles in second-order autonomous system will be analyzed based on the behavior of some appropriate equipotential curves which will be considered around the same limit cycles. In fact two sets of equipotential curves are considered so that a set of the equipotential curves has a role as the upper band of the system trajectories and another set plays a role as the lower band. It will be shown that the stability of the limit cycles in system can be assessed using the behavior of these two set of equipotential curves. It will be shown that asymptotic stability, semi-stability and instability of the limit cycles or oscillation behavior in the system need to analyze both the lower and upper bands set of the equipotential curves. The method can even detect a stable limit cycle appearing in the oscillation systems. The method is geometric and suitable for second-order nonlinear autonomous systems. Finally, some examples will be presented to verify the presented method.**

***Index Terms*—Limit cycle, geometric, stability, nonlinear.**

I. INTRODUCTION

As we know, two basic methods are essentially used to analyze the behavior of limit cycles in nonlinear system [1], [2]. The first method is to draw the trajectories of the system using the softwares such as Matlab in order to detect the limit cycles in the system. It is clear that the limit cycles detected in this approach can be recognized as stable, unstable or semi-stable [1], [3]. The second method is based on the linearization of the system around its equilibrium point or points. The second method has two major weaknesses. Firstly, if the equilibrium point of the system around which the system is linearized is on the limit cycle or in a small neighbor of it, the method may be effective otherwise the method can not detect the real behavior of the nonlinear system. Secondly, the method can only assess the behavior of the system around the limit cycle as the form of point to point if and only if these points all locate on the limit cycle [1]. In other word, all equilibrium points have to locate on the limit cycle and as we know this case is very seldom to happen. The proposed method mentioned in this paper is a geometric method which is more suitable for nonlinear autonomous systems. This kind of systems is very important because it models the behavior of some devices such as oscillators [4]. There are some researches proposing geometric method to design appropriate second-order nonlinear autonomous systems in order to realize and manufacture devices such as oscillators [3]. Also there are some researches presenting geometric methods to analyze the stability and behavior of some systems [5] - [9].

Manuscript received April 6, 2012; revised May 25, 2012.

H. Fathabadi is Post-Doc researcher at National Technical University of Athens, Athens, GREECE. He is also with the Kharazmi University, Tehran, IRAN (e-mail: h4477@ hotmail.com).

II. INVARIANT SET AND EQUIPOTENTIAL CURVES

Consider the nonlinear autonomous system described by the following equations

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \tag{1}$$

Definition 1: A set is said "compact" if it is bounded and closed [1], [3].

Definition 2: Consider the set called P so that $P \subset R^2$, P is said "invariant set" if the trajectories of the system beginning in the P remain in it as $t \longrightarrow \infty$ [1], [10], [11].

Definition 3: Suppose that the $X = 0$ is the equilibrium point of the second-order autonomous system described b y the equation (1) and suppose that the compact set called M includes the equilibrium point (the origin). The closed curves belonging to M, which are described by $u(x_1, x_2) = C$ so that $C \in R$, and enclosing the equilibrium point are called equipotential curves because for each value of C there is a closed curve with the potential of C, so all points locating on the $u(x_1, x_2) = C$ have the equal potential the numerical quantity of which is C [3].

III. LIMIT CYCLE AND STABILITY ANALYSIS

Definition 4: A limit cycle is said asymptotic stable if all trajectories in vicinity of the limit cycle converge to it as $t \longrightarrow \infty$. Otherwise the limit cycle is said semi-stable or unstable [1], [11].

Theorem 1: Consider the second-order autonomous system (1), suppose that no equilibrium point belongs to the compact set M which encloses the origin ($X = 0$). Furthermore assume that there are equipotential curves $u_1(x_1, x_2) = C_1$, and $u_2(x_1, x_2) = C_2$ with clockwise directions, enclosing the origin and satisfying the following inequalities

$$\frac{du_1(x_1, x_2)}{dt} > 0 \tag{2}$$

$$\frac{du_2(x_1, x_2)}{dt} < 0. \tag{3}$$

Then there exists an asymptotic stable limit cycle L so that

$$L \subset \text{int}\,\Omega \tag{4}$$

where $\Omega(C_1, C_{2)}$ is the region located between $u_1(x_1, x_2) = C_1$ and $u_2(x_1, x_2) = C_2$.

$$\frac{du_1(x_1,x_2)}{dx_1}dx_1 + \frac{du_1(x_1,x_2)}{dx_2}dx_2 = 0 \quad (5)$$

Proof: From $u_1(x_1, x_2) = C_1$, we have and as a result, the dynamic of $u_1(x_1, x_2) = C_1$ can be expressed as

$$\begin{cases} \dot{x}_1 = \dfrac{du_1(x_1,x_2)}{dx_2} \\ \dot{x}_2 = -\dfrac{du_1(x_1,x_2)}{dx_1} \end{cases}. \quad (6)$$

Consider the velocity vector on the $u_1(x_1, x_2) = C_1$ symbolized by $\vec{V}_{u_1}$, from (6) we obtain that

$$\vec{V}_{u_1} = \frac{du_1(x_1,x_2)}{dx_2}\vec{u}_{x_1} + (-\frac{du_1(x_1,x_2)}{dx_1})\vec{u}_{x_2} \quad (7)$$

where $\vec{u}_{x_1}$ and $\vec{u}_{x_2}$ are the unique vectors of the x_1 axis and x_2 axis respectively. Also, the velocity vector of the system (1) symbolized by $\vec{\dot{X}}$ can be obtained as

$$\vec{\dot{X}} = f_1(x_1,x_2)\ \vec{u}_{x_1} + f_2(x_1,x_2)\ \vec{u}_{x_2}. \quad (8)$$

The partial derivative $\frac{du_1(x_1,x_2)}{dt}$ can be expressed asand the right hand of above equation can be written as

$$\frac{du_1(x_1,x_2)}{dt} = \frac{du_1(x_1,x_2)}{dx_1}\dot{x}_1 + \frac{du_1(x_1,x_2)}{dx_2}\dot{x}_2 \quad (9)$$

$$\frac{du_1(x_1,x_2)}{dx_1}f_1(x_1,x_2) + \frac{du_1(x_1,x_2)}{dx_2}f_2(x_1,x_2) = \overline{\vec{V}_{u_1} \times \vec{\dot{X}}} \quad (10)$$

where the right hand of the above equation is the algebraic value of the vector product. ,So the inequality (2) can be written as and this means that the direction of the trajectories of the system (1) are to the outside of the equipotential curves $u_1(x_1, x_2) = C_1$ as shown in Fig. 1. In the similar manner the inequality (3) can be expressed as

$$\overline{\vec{V}_{u_1} \times \vec{\dot{X}}} > 0 \quad (11)$$

$$\overline{\vec{V}_{u_2} \times \vec{\dot{X}}} < 0 \quad (12)$$

where $\vec{V}_{u_2}$ is the velocity vector on the $u_2(x_1, x_2) = C_2$ and this means that the direction of the trajectories of the system (1) are to the inside of the equipotential curves $u_1(x_1, x_2) = C_1$ as shown in Fig. 1. On the other hand there is no equilibrium points belonging to M and consequently to $\Omega(C_1, C_2)$, so there is an asymptotic stable limit cycle L so that $L \subset \operatorname{int}\Omega$.□

Example 1: Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 - x_1^{\,7}(x_1^{\,4} + 2x_2^{\,2} - 10) \\ \dot{x}_2 = -x_1^{\,3} - 3x_2^{\,2}(x_1^{\,4} + 2x_2^{\,2} - 10) \end{cases} \quad (13)$$

By choosing, for $0 < C_1 < 2.5$, it can be seen that not only the equipotential curves $u_1(x_1, x_2) = C_1$ are closed but also $\frac{du_1(x_1,x_2)}{dt} > 0$. Also, by choosing

$$u_1(x_1,x_2) = \frac{1}{4}x_1^{\,4} + \frac{1}{2}x_2^{\,2} = C_1$$

$$u_2(x_1,x_2) = \frac{1}{4}x_1^{\,4} + \frac{1}{2}x_2^{\,2} = C_2$$

, for $2.5 < C_2$, it can be seen that not only the equipotential curves $u_2(x_1, x_2) = C_2$ are closed but also $\frac{du_2(x_1,x_2)}{dt} < 0$, so there is an asymptotic stable limit cycle locating between the $u_1(x_1, x_2) = C_1$ and $u_2(x_1, x_2) = C_2$. The area locating between $u_1(x_1, x_2) = C_1$ and $u_2(x_1, x_2) = C_2$ is an invariant set expressed by the following set

$$\Omega(C_1, C_2) = \left\{(x_1, x_2) \middle| C_1 < \frac{1}{4}x_1^{\,4} + \frac{1}{2}x_2^{\,2} < C_2\right\} \quad (14)$$

So that $0 < C_1 < 2.5\ \ and\ \ C_2 > 2.5$.It is clear that the limit cycle can be estimated by varying C_1 and C_2 in (14). In above invariant set, by increasing C_1 and decreasing C_2, the asymptotic stable limit cycle can be earned as

$$\frac{1}{4}x_1^{\,4} + \frac{1}{2}x_2^{\,2} = 2.5$$

IV. Control of Limit Cycle Using State Feedback

Consider the nonlinear autonomous system described by the following equations

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u_1^*) \\ \dot{x}_2 = f_2(x_1, x_2, u_2^*) \end{cases} \quad (15)$$

where u_1^* and u_2^* are the control inputs as the form of state feedback presented by the following equations

$$\begin{cases} u_1^* = h_1(x_1, x_2) \\ u_2^* = h_2(x_1, x_2) \end{cases}. \quad (16)$$

Now, the question is that how $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ must be chosen so that an asymptotic stable limit cycle can be added to the system (15)?

Using (9), the condition $\frac{du_1(x_1,x_2)}{dt} > 0$ of the theorem (1) can be earned as the following inequalityand in the similar manner the $\frac{du_2(x_1,x_2)}{dt} < 0$, which appeared in theorem (1), can be expressed as

$$\frac{du_1(x_1,x_2)}{dx_1} f_1(x_1,x_2,u_1^*) + \frac{du_1(x_1,x_2)}{dx_2} f_2(x_1,x_2,u_2^*) > 0 \quad (17)$$

$$\frac{du_2(x_1,x_2)}{dx_1} f_1(x_1,x_2,u_1^*) + \frac{du_2(x_1,x_2)}{dx_2} f_2(x_1,x_2,u_2^*) < 0. \quad (18)$$

The equations (17) and (18) give the conditions which have to be satisfied by u_1^*, u_2^*, $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in order to appear an asymptotic limit cycle in the system (15).

Example 2: Consider the following system

$$\begin{cases} \dot{x}_1 = x_2^7 - x_1^3 + u_1^* \\ \dot{x}_2 = -x_1 - x_1^2 x_2 + u_2^* \end{cases} \quad (19)$$

It is clear that the equilibrium point at the origin is asymptotic stable. Now, the state feedback lows (u_1^* and u_2^*) have to be determined so that an asymptotic stable limit cycle can be added to the resulted closed loop system. By choosing equipotential curves asand replacing (20) and (21) in (17) and (18) respectively, the following inequalities are earned

$$u_1(x_1, x_2) = 4x_1^2 + x_2^8 = C_1;\ 0 < C_1 < 12 \quad (20)$$

$$u_2(x_1, x_2) = 4x_1^2 + x_2^8 = C_2;\ 14 < C_2 \quad (21)$$

$$8x_1(x_2^7 - x_1^3 + u_1^*) + 8x_2^7(-x_1 - x_1^2 x_2 + u_2^*) > 0;$$

for $0 < 4x_1^2 + x_2^8 < 12$ (22)

and

$8x_1(x_2^7 - x_1^3 + u_1^*) + 8x_2^7(-x_1 - x_1^2 x_2 + u_2^*) < 0$; for $14 < 4x_1^2 + x_2^8$. (23)

It can be derived from (22) and (23) that

$$-8x_1^4 - 8x_1^2 x_2^8 + 8x_1 u_1^* + 8x_2^7 u_2^* > 0;$$

for $0 < 4x_1^2 + x_2^8 < 12$ (24)

and

$$-8x_1^4 - 8x_1^2 x_2^8 + 8x_1 u_1^* + 8x_2^7 u_2^* < 0;$$

for $14 < 4x_1^2 + x_2^8$. (25)

By choosing the state feedback lows as the form of the following equations and replacing in (24) and (25), the following inequalities are earned as the conditions that must be satisfied in order to appear an asymptotic limit cycle in the system (15)

$$\begin{cases} u_1^* = h_1(x_1, x_2) = x_1(\beta + \frac{3}{4} x_2^8) \\ u_2^* = h_2(x_1, x_2) = 0 \end{cases} \quad (26)$$

$$-2x_1^2(4x_1^2 + x_2^8 - 4\beta) > 0;$$

For $0 < 4x_1^2 + x_2^8 < 12$ (27)

and

$$-2x_1^2(4x_1^2 + x_2^8 - 4\beta) < 0;$$

for $14 < 4x_1^2 + x_2^8$. (28)

The inequalities (27) and (28) both are satisfied, when

$$3 \le \beta \le \frac{14}{3}. \quad (29)$$

It also follows from the theorem 1 that the asymptotic stable limit cycle called L and added to the system (15) using state feedback appears in the following region

$$L \subset \{(x_1, x_2) \mid 12 \le 4x_1^2 + x_2^8 \le 14\}. \quad (30)$$

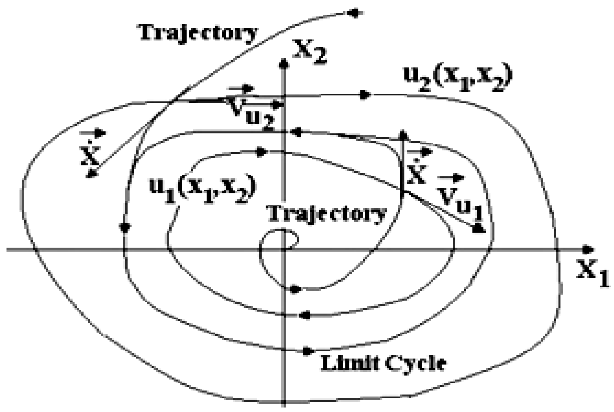

Fig. 1. The system has an asymptotic stable limit cycle.

V. Conclusions

The method presented in this paper was a geometric method which is suitable for nonlinear autonomous systems. It can be seen that the equipotential curves presented in this paper are the same energy functions presented by lyapunov to analyze the stability of the equilibrium points in systems. In fact the innovation of this paper is to present a geometric method to recognize the limit cycles of the nonlinear autonomous systems.

References

[1] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd edition, Prentice Hall: Englewood Cliffs, NJ, 1993.

[2] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*, Macmillan: New York, 1992.

[3] H. Fathabadi and S. K. Nikravesh, "A Theoretical method for design and realization of fixed amplitude sinusoidal oscillators," *Analog* Integrated *Circuits and Signal Processing*, Kluwer Academic Publishers, vol. 39, pp. 123-131, 2004.

[4] P. E. Gray, P. J. Hurst, S. H. Lewis, and R. G. Meyer, *Analysis and Design of Analog Integrated Circuits*, Wiley: New York, 2001.

[5] N. Xie and S. Weber, "Geometric approximations of some Aloha-like stability regions," in *Proc. IEEE Int. Symp. On Information Theory Proceedings (ISIT)*, pp. 1848-1852, 2010.

[6] M. P. Mann and Z. Shiller, "Dynamic stability of off-road vehicles: A geometric approach," in *Proc*. *IEEE Int. Conf. on Robotics and* Automation *(ICRA 2006)*, pp. 3705-3710, doi:10.1109/ROBOT.2006.1642268, 2006.

[7] S. Benkler, N. Chavannes, and N. Kuster, "A new 3-D conformal PEC FDTD scheme with user-defined geometric precision and derived stability criterion," *IEEE Trans. on Antennas and Propagation*, vol. 54, pp. 1843-1849, 2006.

[8] G. Marro and E. Zattoni, "A novel geometric insight into the model matching problem with stability," in *Proc. 41st IEEE Conf. on Decision and Control,* vol. 2, pp. 2137-2142, 2002.

[9] L. Ntogramatzidis, M. Cantoni, and R. Yang, "A geometric approach with stability for two-dimensional systems," in *Proc. 46th IEEE Conf. on* Decision *and Control*, pp. 4391-4396, doi:10.1109/CDC.2007.4434539, 2007.

[10] H. Nijmerijer and A. J. Vander Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, 1990.

[11] B. Friedland, *Advanced Control System Design*, Prentice Hall: Englewood Cliffs, NJ, 1996.