L-Moment and Inverse Moment Estimation of the Inverse Generalized Exponential Distribution

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Abstract—This paper presents the L-Moment and Inverse Moment estimation of the inverse Generalized Exponential distribution also the properties of the L-Moment. The main interests are in the relationship between $\beta$ and various L-Moment; measure of variability for L-Moment as the numerical quantities that describe the spread of the values in a set of data. Here these L-Moment models presented graphically and mathematically. The Inverse Moment estimation models are MTTF, Coefficient of Variation, Coefficient of Skewness and Coefficient of kurtosis are presented mathematically.

Index Terms—Inverse Generalized Exponential distribution, l-moments, l-moments estimation, l-coefficient of variation, l-skewness, l-kurtosis, inverse moment estimation.

I. INTRODUCTION

Recently a new two parameter distribution, named as Inverse Generalized Exponential (IGE) probability distribution has been introduced by the author M. Shuaib Khan [1], [2], [3], [4]. The Inverse Generalized Exponential probability distribution is the newly developed probability distribution has two parameters $\beta$ and $\eta$. It can be used to represent the failure probability density function (PDF) and is given by

$$f_{\text{IGE}}(t) = \frac{\beta}{\eta t - \theta} e^{-\frac{1}{\eta t - \theta}}$$

and the corresponding cdf is

$$F_{\text{IGE}}(t) = 1 - e^{-\left(\frac{t}{\eta} - \frac{1}{\eta t - \theta}\right)}$$

where $\beta$ is the shape parameter representing the different pattern of the Inverse Generalized Exponential PDF and is positive and $\eta$ is a scale parameter representing the characteristic life at which 96.8% of the population can be expected to have failed and is also positive, $t_0$ is a location or shift or threshold parameter (sometimes called a guarantee time, failure-free time or minimum life). It is important to note that the restrictions in eq. (1.1) on the values of $t_0, \eta, \beta$ are always the same for the Inverse Generalized Exponential distribution.

Fig. 1.1 shows the diverse shape of the Inverse Generalized Exponential PDF with $t_0 = 0$ and value of $\eta = 1$ and $\beta = (0.8, 1, 1.5, 2, 2.5)$.

II. L-MOMENTS

As discussed in previous section, the alternative measure of distribution of shapes denoted by Hosking [5][6], L-moments are expectation of certain linear combinations of order Statistics. Hosking has defined the L-moments of $X$ to be the quantities.

$$\lambda_r = \frac{1}{r!} \sum_{k=0}^{\infty} (-1)^k \binom{r-1}{k} E(X_{r-k}; \theta)$$

L in “L-moments” emphasizes that $\lambda_r$ is a linear function of the expected order statistics. The expectation of an order Statistics has been written as Hosking [5]

$$E(X_{r-k}; \theta) = \frac{1}{(j-1)!(r-j)!} \int x(F)[F(x)]^{j-1}[1-F(x)]^{r-j} dF(x).$$

The first few L-moments, $\lambda_r$ of random variable "X", as defined by Hosking [1] are given below:

$$\lambda_1 = E(X) = \frac{1}{\theta} \int x(F)dF$$

$$\lambda_2 = \frac{1}{2} E(\frac{X_2^2}{2} - X_1^2) = \frac{1}{\theta} \int x(F)(2F - 1)dF$$

$$\lambda_3 = \frac{1}{3} E(\frac{X_3^3}{3} - 2X_2^2 + X_1^3) = \frac{1}{\theta} \int x(F)(6F^2 - 6F + 1)dF$$
\[ \lambda_4 = \frac{1}{4} E\left[X_4; 4 - 3X_3; 4 + 3X_2; 4 - X_1; 4\right] = \frac{1}{0} x(F\left(20e^3 - 30e^2 + 12F - 1\right))dF \]

where \(X_k; n\) is an order Statistics, the \(k^{th}\) smallest of a sample of size \(n\) drawn from the distribution of \(X\) and \(x(F)\) is a quantile function of real-valued random variable \(X\). The “L” in L-moment emphasized that \(\lambda_r\) is a linear function of expected order Statistics. Measures of skewness and kurtosis, based on L-moments, are respectively.

\[
\text{L-skewness} = \tau_3 = \frac{\lambda_3}{\lambda_2} \tag{2.3}
\]

And

\[
\text{L-Kurtosis} = \tau_4 = \frac{\lambda_4}{\lambda_2} \tag{2.4}
\]

The L-moments \(\lambda_1, \cdots, \lambda_r\) and L-moment ratios \(\tau_3, \cdots, \tau_r\) are useful quantities for summarizing a distribution. The L-moments are in some ways analogous to the (conventional) center moments and the L-moments ratios are analogous to moment’s ratios. In particular \(\lambda_1, \lambda_2, \tau_3\) and \(\tau_4\) may be regarded as measures of location, scale, skewness and kurtosis respectively. Hosking [5] has shown that L-moments have good properties as measure of distributional shape and are useful for fitting distribution to data.

III. CALCULATION OF L-SKEWNESS AND L-KURTOSIS

The probability distribution can be summarized by the following four measures. The mean or L-location (\(\lambda_1\)), The L-scale (\(\lambda_2\)), The L-skewness (\(\tau_3\)), The L-Kurtosis(\(\tau_4\)), we now consider these measures, particularly \(\tau_3\) and \(\tau_4\) in more details. Moments are often used as summary measure of the shapes of a distribution. In this section the measure of skewness and kurtosis based on L-moments has been calculated from Inverse Generalized Exponential distribution. The first four L-moments of Inverse Generalized Exponential distribution given in expression (1.1) have been calculated from relations (2.2), as shown in fig. (3.1).

\[
\lambda_1 = \frac{\Gamma\left(1 - \frac{1}{\beta}\right)}{\eta} \tag{3.1}
\]

\[
\lambda_2 = \frac{\Gamma\left(1 - \frac{1}{\beta}\right)}{\eta^{\frac{1}{\beta}}} \left[\frac{1}{2^{\frac{1}{\beta}}} - 1\right] \tag{3.2}
\]

\[
\lambda_3 = \frac{\Gamma\left(1 - \frac{1}{\beta}\right)}{\eta} \left[1 - 3(2)\frac{1}{\beta} + 2(3)\frac{1}{\beta}\right] \tag{3.3}
\]

\[
\lambda_4 = \frac{\Gamma\left(1 - \frac{1}{\beta}\right)}{\eta} \left[1 + 6(2)\frac{1}{\beta} - 10(3)\frac{1}{\beta} + 5(4)\frac{1}{\beta}\right] \tag{3.4}
\]

The above function describes the L-location of the inverse generalized exponential distribution. For the graphical analysis L moment of measure of central tendency is the mean denoted by \(\lambda_1\). In the graphical analysis \(\beta\) is the shape parameter and \(\eta\) is a scale parameter representing the characteristic life. Fig. 2.1 shows the oscillate shape of the L-location of the Inverse Generalized Exponential with value of \(\eta = 10\).

The corresponding measures of L-Coefficient of Variation, L-skewness \(\tau_3\) and L-kurtosis \(\tau_4\) using (2.3) and (2.4) are shown in fig. (3.2), (3.3), (3.4).

\[
CV = 2^{\frac{1}{\beta}} - 1 \tag{3.5}
\]

\[
\tau_3 = 1 - 3(2)\frac{1}{\beta} + 2(3)\frac{1}{\beta} \tag{3.6}
\]

The L-C.V is used for checking the consistency of the data shown in Fig. 3.2. The Relationship between \(\beta\) and \(CV_{IGE}\) shows that as \(\beta \rightarrow \infty\) then the value of \(CV_{IGE}\) decreases asymptotically. \(C.SK_{IGE}\) is the quantity used to measure the
skewness of the distribution. The relationship between $\beta$ and $C.SK_{IGE}$ is shown in Fig 3.3. Fig 3.3 shows that as $\beta \to \infty$ then the value of $L - C.SK_{IGE}$ increases asymptotically.

\[
\tau_4 = \frac{1 + 6(2)^\frac{1}{\beta} - 10(3)^\frac{1}{\beta} + 5(4)^\frac{1}{\beta}}{2^\beta - 1}\quad (3.7)
\]

$K_{IGE}$ is the quantity, which can be used to measure the kurtosis. The relationship between $\beta$ and $K_{IGE}$ is shown in Fig 3.4. Here we note that as $\beta \to \infty$ then the values of $K_{IGE}$ is positively skewed has a maximum value asymptotically. Hence $\beta$ and $K_{IGE}$ have a positive proportion when $\beta > 1.1$.

The $L$-moment ratio of Inverse Generalized Exponential diagram can be used to compare the relations between $L$-skewness vs $L$-kurtosis. The $L$-moment ratio diagram shows that as skewness increases then the pattern of the kurtosis become decreases.

### IV. A NUMERICAL ILLUSTRATION

TABLE I: RELATIONSHIP B/W $\beta$ VS L-COEFFICIENT OF VARIATION, $L$-SKEWNESS ($\tau_3$) AND $L$-KURTOSIS ($\tau_4$) FOR INVERSE GENERALIZED EXPONENTIAL DISTRIBUTION

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$L-CV_{IGE}$</th>
<th>$L-CS_{IGE}$</th>
<th>$L-C.K_{IGE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0.414214</td>
<td>0.534654</td>
<td>5.226225</td>
</tr>
<tr>
<td>3</td>
<td>0.259921</td>
<td>0.402953</td>
<td>7.979484</td>
</tr>
</tbody>
</table>
Table (1) presents 50 observations for L-coefficient of variations, L-coefficient of skewness, and L-coefficient of kurtosis by using simulation technique.

From Table (1), one can show that
- The values of \( L - C.V_{IGED} \) decreases asymptotically as \( \beta \to \infty \).
- The values of \( L - C.S_{IGED} \) increases asymptotically as \( \beta \to \infty \).
- The values of \( L - C.K_{IGED} \) increases as the value of \( \beta \) increases.

V. INVERSE MOMENTS ESTIMATION OF INVERSE GENERALIZED EXPONENTIAL DISTRIBUTION

The method of Inverse Moments Estimation of Inverse Generalized Exponential model is defined as
\[
\mu_{r-1} = \frac{\eta}{\beta} \Gamma((1 + r)) \cdot r = 1, 2, 3, 4, \ldots \quad (5.1)
\]

The inverse Moments Estimator is helpful for finding the properties of the Inverse Generalized Exponential distribution. The method of inverse Moments Estimator for the parameters is
\[
m_{r-1} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{t_{si}}
\]

For the convenience of display, we present the (5.1) as
\[
\omega_r = \Gamma(1 + r), \quad r = 1, 2, 3, 4, \ldots
\]

\[
\mu_r = \frac{\eta}{\beta} \omega_r, \quad r = 1, 2, 3, 4, \ldots \quad (5.1a)
\]

For the Rth moment estimation some of the important properties of the Inverse Generalized Exponential distribution are MTTF, Coefficient of variation, coefficient of skewness and coefficient of kurtosis.

The mean life of the Inverse Generalized Exponential distribution is
\[
MTTF_{IGE} = \frac{\eta}{\beta} \omega_1 \quad (5.2)
\]

The coefficient of variation \( CV_{IGE} \) life of the Inverse Generalized Exponential distribution is defined as
\[
SD_{IGE} = \sqrt{\frac{\omega_2}{\omega_1^2}} - 1 \quad (5.3)
\]

From my calculation it is clear that there is unit coefficient of variation life \( CV_{IGE} = 1 \). The relationship between \( \beta \) and \( CV_{IGE} \) is fixed. The larger or the smaller the value of \( \beta \) has no effect on the value of \( CV_{IGE} \).

The coefficient of skewness \( CS_{IGE} \) life of the Inverse Generalized Exponential distribution is defined as
\[
CS_{IGE} = \frac{E\left(t - \frac{\beta}{\eta} \omega_1\right)^3}{\left(\frac{\eta^2}{\beta^2} (\omega_2 - \omega_1^2)\right)^{3/2}} \quad (5.4)
\]

where \( CS_{IGE} = 2 \) is the quantity used to measure the skewness of the Inverse Generalized Exponential distribution, here \( CS_{IGE} > 0 \) so the PDF of the Inverse Generalized Exponential distribution is skewed to the right when (Mean > Median > Mode). The relationship between \( \beta \) and \( CS_{IGE} \) is a fix value. From my calculatios it is clear that there is no \( CS_{IGE} \) life when \( \beta < 0 \).

The coefficient of kurtosis \( CK_{IGE} \) life of the Inverse Generalized Exponential distribution is defined as
\[
CK_{IGE} = \frac{E\left(t - \frac{\beta}{\eta} \omega_1\right)^4}{\left(\frac{\eta^2}{\beta^2} (\omega_2 - \omega_1^2)\right)^2} \quad (5.5)
\]

where \( CK_{IGE} = 9 \) the quantity is used to measure the kurtosis or peaked ness of the distribution. The Inverse Generalized exponential PDF shape is more peaked than the Normal PDF because the value of \( CK_{IGE} \geq 3 \).

VI. SUMMARY AND CONCLUSION

In this paper the quantile comprehensive study of the Inverse Generalized Exponential quantile modeling is predicted for finding the life time of the electrical and mechanical components. These patterns of \( \beta \) and various B-lives are helpful for finding the life of components. In this paper we also presented measure of variability for B-lives as the numerical quantities that describe the spread of the values in a set of data. Here the inverse Moments Estimator is helpful for finding the properties of the Inverse Generalized Exponential distribution.

REFERENCES
Mr. Muhammad Shuaib Khan was born in Multan, Pakistan in 1977. He has M.Phil in Statistics, M.Sc in Statistics and (PGD) Post graduate Diploma in computer programming and computing Statistics from Bahaudin Zakaria University Multan Pakistan. He is working as a Lecturer in the Department of Statistics, The Islamia University of Bahawalpur, Pakistan. He has 35 research publications from which 22 journals publication and 13 included in conference-related papers. He has about 12 Years of research and teaching experience at undergraduate, postgraduate and research level students.

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