Delay Estimation for Global RC Interconnect Using Inverse Gamma Distribution Function

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***Abstract*—In this paper, we present an efficient model for on-chip interconnects delay analysis. Several approaches have already been proposed for approximating the interconnect delay accurately and efficiently. Moments of the impulse response are widely used for interconnect delay analysis, from the explicit Elmore delay (the first moment of the impulse response) expression to the moment matching methods which create reduced order trans-impedance and transfer function approximations. However, the Elmore delay is fast becoming ineffective for deep submicron technologies and reduced order transfer function delays are impractical for use as early-phase design metrics or as design optimization cost functions. This paper describes an accurate approach for fitting moments of the impulse response to probability density functions so that delay can be estimated accurately at an early physical design stage. For RC trees, it is demonstrated that inverse gamma functions provide a probably stable approximation. The accuracy of the proposed model is justified with the results obtained from that of SPICE simulations.**

***Index Terms*—Delay Calculation, inverse gamma distribution, moment matching, on-chip interconnect; probability distribution function.**

I. Introduction

As integrated circuit feature sizes continue to scale well below 0.18 microns, active device counts are reaching hundreds of millions [1]. The advent of sub-quarter-micron IC technologies has forced dramatic changes in the design and manufacturing methodologies for integrated circuits and systems. The paradigm shift for interconnect which was once considered just a parasitic, can now be the dominant factor to determine the integrated circuit performances. It results the greatest impetus for change of existing methodologies. Over the past decade there have been a number of advances in modeling and analysis of interconnect that have facilitated the continual advances in design automation for systems of increasing frequency and downsizing.

The amount of interconnect among the devices tends to grow super linearly with the transistor counts, and the chip area is often limited by the physical interconnect area. Due to these interconnect area limitations, the interconnect dimensions are scaled with the devices whenever possible. In addition, to provide more are scaled with the devices whenever possible. In addition, to provide more wiring resources, IC's now accommodate numerous metallization layers, with more to come in the future. These advances in technology that result in scaled, multi-level interconnects may address the wireability problem, but in the process creates problems with signal integrity and interconnect delay. This paper proposes an extension of Elmore's approximation [2] to include matching of higher order moments of the probability density function. Specifically, using a time-shifted incomplete gamma function approximation [3] for the impulse responses of RC trees, the three parameters of this model are fitted by matching the first three central moments (mean, variance, skewness), which is equivalent to matching the first two moments of the circuit response (m_1, m_2,). Importantly, it is proved that such an inverse gamma fit is guaranteed to be realizable and stable for the moments of an RC tree. Once the moments are fitted to characterize the inverse gamma function, the step response delay is obtained as a closed form expression thereby providing the same explicitness as that of the Elmore approximation. This work is simple yet accurate compared to the model proposed in the respect that our approach provides a closed form expression and does not require any look up table to calculate the delay for RC interconnect.

The rest of the paper is organized as follows: Section 2 describes the basic theory, expressions of circuit moments in terms of impulse response and expressions of mean and variance in terms of circuit moments. Section 3 discusses the proposed delay metric based on inverse gamma distribution and the final expression for interconnect delay in terms of first two circuit moments. Section 4 shows the simulation results and the comparison with other established matrices. Finally Section 5 concludes the paper

II. Basic Theory

A. Moments of a Linear Circuit Response

Let $h(t)$ be a circuit impulse response in the time domain and let H(s) be the corresponding transfer function. By definition, H(s) is the Laplace transform of h (t) [4],

$$H(s) = \int_0^\infty h(t) e^{-st} dt \tag{1}$$

Applying a Taylor series expansion of e^{-st} about s = 0, yields the transfer function.

Then the transfer function is given by:

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$$H(s)=\left(\frac{\lambda}{\lambda+s}\right)^n \quad (2)$$

$$H(s)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}s^i\int_0^{\infty}t^i h(t)dt \quad (3)$$

The i^{th} circuit-response moment, $\widetilde{m}_i$ is defined as [5]:

$$\widetilde{m}_i=\frac{(-1)^i}{i!}\int_0^{\infty}t^i h(t)dt \quad (4)$$

From (3) and (4), the transfer function H(s) can be expressed as:

$$H(s)=\widetilde{m}_0+\widetilde{m}_1 s+\widetilde{m}_2 s^2+\widetilde{m}_3 s^3+... \quad (5)$$

B. Central Moments

It is straightforward to show that the first few central moments can be expressed in terms of circuit moments as follows [6]:

$$\left.\begin{aligned}&\mu_0=m_0,\mu_1=0,\mu_2=2m_2-\frac{m_1^2}{m_0},\\&\mu_3=-6m_3+6\frac{m_1 m_2}{m_0}-2\frac{m_1^3}{m_0^2}\end{aligned}\right\} \quad (6)$$

Unlike the moments of the impulse response, the central moments have geometrical interpretations:

μ_0 is the area under the curve. It is generally unity, or else a simple scaling factor is applied.

μ_2 is the variance of the distribution which measures the spread or the dispersion of the curve from the center. A larger variance reflects a larger spread of the curve.

μ_3 is a measure of the skewness of the distribution; for a unimodal function, its sign determines whether the mode (global maximum) is to the left or to the right of the expected value (mean). Its magnitude is a measure of the distance between the mode and the mean.

C. Higher Central Moments in RC Trees

The second and third central moments are always positive for RC tree impulse responses [6]. The positiveness of the second order central moment is obvious from its definition;

$$\mu_2=\int_0^{\infty}(t-\mu)^2 h(t)dt \quad (7)$$

The impulse response, $h(t)$, at any node in an RC tree is always positive. Hence the second central moment μ_2 is always positive.

D. Moments of Probability Density Functions

A probability function is a real valued set function where the domain is a subset of the sample space, *S*, and the range is a real number in the interval [0, 1]. Generally, a function Pr {*} should satisfy the three Kolmogorov axioms [7], or equivalent conditions, in order to be considered a probability functions:

1) $\Pr\{S\}=1$;
2) $\Pr\{A\}\geq 0$ *for all* $A\in S$;
3) $\Pr\{A\cup B\}=\Pr\{A\}+\Pr\{B\}$ *if* $A\cap B=\phi, A\in S, B\in S$.

The distribution function of a continuous random variable T denoted as $F_T(t)$ provides the value of Pr {T≤ t} for any real number $-\alpha\leq t\leq\alpha$. The associated probability density function (pdf) denoted as $f_T(t)$ is the derivative of the distribution function with respect to *t*, thus,

$$f_T(t)=\frac{dF_T(t)}{dt} \quad (8)$$

$$F_T(t)=\int_{\alpha}^{t}f_T(\tau)d\tau \quad (9)$$

The median, $t_{(0.5)}$, is defined by,

$$P_r\{T\leq t_{0.5}\}=F_T(t_{(0.5)})=\int_{-\alpha}^{t_{(0.5)}}f_T(t)dt=0.5 \quad (10)$$

Whereas, the expected value or mean, $E[T]$ of a continuous random variable *T* with distribution $f_T(t)$ is,

$$E[T]=\int_{-\alpha}^{\alpha}tf_T(t)dt$$

The mean is also the first moment of the distribution. In General, the i^{th} moment m_i of the distribution is

$$m_i=E[T^i]=\int_{-\alpha}^{\alpha}t^i f_T(t)dt \quad (11)$$

E. Relation between Probability Density Functions and Circuit Responses

Any function f(t) can be treated as a probability density function if it is defined in the range [a, b] and satisfies

$$\left.\begin{aligned}&f(t)\geq 0\forall t\\&\int_a^b f(t)dt=1\end{aligned}\right\} \quad (12)$$

If *f*(*t*) is equal to zero outside of the range [a, b], we can replace the integration limits in (11) with -∞ and ∞. Elmore [2] was the first to apply moments for delay approximation of a limited class of circuit responses by observing that the impulse response of a circuit can be treated as a probability density function. Elmore model used this observation to justify the approximation of the 50% point of a monotonic step response (the median point of the impulse response) by the first moment (mean of the impulse response). It was shown that the impulse response corresponding to an RC tree is unimodal with positive skew [6]. From this it follows that the mode is less than the median which is less than the mean and vice versa [8] [9].

(Skew > 0) if and only if (mode <median < mean)

This proved that the Elmore delay [2] is an upper bound for the 50% step response delay, and was shown to hold for finite input signal rise time. An important observation [6] is that because of the variation in impulse response shapes along an interconnect path, the relative accuracy of the Elmore delay bound can be quite poor. Especially for the interconnects associated with deep submicron technologies, more than one moment is needed to capture the waveform shape characteristics.

III. Proposed Delay Model

Elmore's original delay approximation is based on the analogy between non-negative impulse responses and probability density functions. In theory, Elmore's distribution interpretation can be extended beyond simply estimating the median by the mean if higher order moments can be used to characterize a representative distribution function. Once characterized, the delay can be approximated via closed form expression or table lookup of the median value for the representative distribution family. The proposed model is based on the inverse gamma distribution. The inverse gamma distribution is a two parameter continuous distribution [10]. It is well suited to match the impulse response of the generalized RC network since both are unimodel and have non-negative skewness. The inverse gamma distribution's probability density function is defined over $x > 0$

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\left(\frac{1}{x}\right)^{\alpha+1} \exp\left(-\frac{\beta}{x}\right) \quad (13)$$

with shape parameter *α* and scale parameter *β*.

Mean for the inverse gamma distribution is given by,

$$Mean(E[x]) = \frac{\beta}{\alpha-1} \quad (14)$$

Variance is given by,

$$Variance = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad (15)$$

Mode is given by,

$$Mode = \frac{\beta}{\alpha+1} \quad (16)$$

A. Calculation for the parameters α and β

Since inverse gamma distribution has two parameters *α* and *β*, so by using moment generating function completely characterizes this model. Hence the parameters *α* and *β* of inverse gamma distribution can be presented in terms of moments. Moment generating function of inverse gamma distribution is given as bellow:

$$E[X^n] = \frac{\beta^n}{(\alpha-1)(\alpha-2)\ldots\ldots\ldots\ldots(\alpha-n)} \quad (17)$$

When n=1,

$$E[X] = \frac{\beta}{\alpha-1} \quad (18)$$

When n=2,

$$E[X^2] = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad (19)$$

Hence we can write,

$$m_1 = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad (20)$$

$$m_2 = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad (21)$$

From (20) and (21),

$$(\alpha^2 - 3\alpha + 2)m_2 = (\alpha-1)^2 m_1^2 \quad (22)$$

After solving above equation,

$$\alpha^2(m_2 - m_1^2) + \alpha(2m_1^2 - 3m_2) + (m_2 - m_1^2) = 0 \quad (23)$$

By solving (23) we get,

$$\alpha = \frac{-(2m_1^2 - 3m_2) \pm \sqrt{(2m_1^2 - 3m_2) - 4(m_2 - m_1^2)(2m_2 - m_1^2)}}{4(m_2 - m_1^2)} \quad (24)$$

By taking Positive sign we get for α<1/2

$$\alpha = \frac{4m_2 - 2m_1^2}{4(m_2 - m_1^2)} \quad (25)$$

From (20) and (25), we can write for *β* as,

$$\beta = \frac{2m_1^3}{4(m_2 - m_1^2)} \quad (26)$$

Calculation of Median (50% Delay)

The Median of the inverse gamma distribution is defined as,

$$Median = \frac{1}{3}[Mode + 2Mean] \quad (27)$$

Substituting (14) and (16) in (27) we get

$$Median = \frac{1}{3}\left[\frac{2\beta}{\alpha-1} + \frac{\beta}{\alpha+1}\right] \quad (28)$$

After solving,

$$Median = \frac{(3\alpha+1)\beta}{3(\alpha+1)(\alpha-1)} \quad (29)$$

Now substituting the values of *α* and *β* from (25) and (26) in (29), we have,

$$Median = \left(\frac{\left(3\frac{2(m_2 - m_1^2)}{4(m_2 - m_1^2)} + 1\right)\frac{2m_1^3}{4(m_2 - m_1^2)}}{3\left(\frac{2(m_2 - m_1^2)}{4(m_2 - m_1^2)} + 1\right)\left(\frac{2(m_2 - m_1^2)}{4(m_2 - m_1^2)} - 1\right)}\right) \quad (30)$$

After simplification of the above equation, we get,

$$50\%\, Delay = Median = \frac{m_1(8m_2 - 5m_1^2)}{3(4m_2 - 3m_1^2)} \quad (31)$$

The above equation (31) is the proposed closed form delay expression for generalized RC Global interconnects using inverse gamma distribution based on moment generating technique. From the equation we can see that the delay expression is merely simple function of first two circuit moments of the system.

IV. Simulation Results

Simulation results for general tree structures of RC interconnect are presented in this section to verify the validity of the proposed method. We have implemented the proposed delay estimation method using inverse gamma distribution and applied it to widely used actual interconnect RC networks as shown in Fig. 1. For each RC network source we place a driver, where the driver is a voltage source followed by a resister.

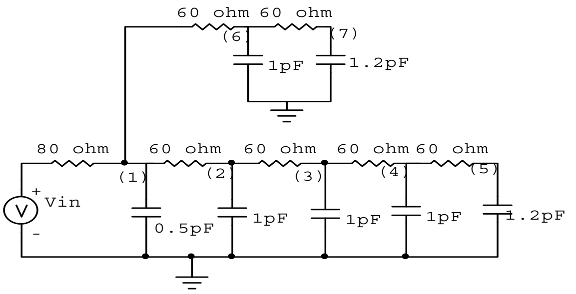

Fig. 1. A RC Tree Example

We compared the delays obtained from SPICE, Weibull Distribution (WD) [11] and Gamma Distribution (GD) [12] with those found using the inverse gamma approximation (IGD). The results for the 50% delay are summarized in Table I.

TABLE I: COMPARISON OF THE 50% DELAY (TIME IN NS)

Node	SPICE (50%)	GD (50%)	(WD) (50%)	IGD (50%)
1	0.196	0.234	0.317	0.217
2	0.477	0.493	0.412	0.462
3	0.700	0.697	0.594	0.587
4	0.845	0.828	0.817	0.837
5	0.919	0.923	0.949	0.929
6	0.375	0.373	0.351	0.363
7	0.452	0.451	0.442	0.447

Note that the difference between the inverse gamma delay (IGD) obtained by the proposed technique and the SPICE delay at the leaf nodes is about 2%. The worst case delay obtained for the nodes that are closest to the node 3. These nodes correspond to the responses with the highest frequency element, therefore one would expect the largest moment matching error.

This technique requires a bell-shaped distribution for the impulse response, over damped and critically damped RC circuit responses. The impulse response at the driving point node does not follow the bell shape, but rather starts out with a non-zero value and asymptotically approach zero in a multi-exponential decay form. Other distribution families may permit matching higher order moments, or more naturally capture these driving point response shapes. But as with all moment matching problems, the greatest challenge is to find a model that is provably stable and realizable.

V. CONCLUSIONS

We have proposed Inverse Gamma-Distribution function based closed form delay model for the RC trees that is a simple function of two moments of impulse response. In this paper the probability interpretation is extended to the circuit homogeneous response. For a generalized RC interconnect model the stability of the homogeneous Inverse gamma model is verified on the industrial nets.

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