Improved Results on Passivity Analysis of Neutral-Type Neural Networks with Mixed Time-Varying Delays

T. Botmart, N. Yotha, K. Mukdasai, and W. Weera

Abstract—**This paper is considered with problem of the passivity analysis for neutral-type neural networks with mixed time-varying delays. By constructing an augmented Lyapunov -Krasovskii functionals and using the double integral inequality with approach to estimate the derivative of the Lyapunov-Krasovskii functionals, sufficient conditions are established to ensure the passivity of the considered neutral-type neural networks, in which some useful information on the neuron activation function ignored in the existing literature is taken in to account. Finally, a numerical example is given to demonstrate the effectiveness of the proposed method.**

Index Terms—**Passivity, neutral-type neural networks, time delay, integral inequality.**

I. Introduction

In the past few decades, delayed neural networks (NNs) have been an important issue due to their applications in many areas such as signal processing, pattern recognition, associative memories, fixed-point, computations, parallel computation, control theory and optimization solvers [1]-[3]. The state estimation problems for NNs with discrete interval and distributed time-varying delays have been extensively studied in [4]-[6]. On the other hand, it is common that the time delay of system state, have been many phenomenon such as automatic control, chemical reactors, distributed networks, heat exchanges, etc. The systems containing the information of past state derivatives are called neutral-type neural networks (NTNNs). The existing work on the state estimator of NTNNs with mixed delays are only [7]-[8] at present. In [9], the authors considered the problem of global passivity analysis of interval neural networks with discrete and distributed delays of neutral type. Consequently, the passivity analysis of NTNNs has also been received considerable attention and lots of works were reported in recent years. The problem of passivity performance analysis has also been extensively applied in many areas such as signal processing, sliding mode control, and networked control [10]-[12]. The main idea of the passivity theory is that the passive properties of a system can keep the system internally stable. In [13]-[17], authors investigated the passivity of neural networks with time-varying delay, and gave some criteria for checking the passivity of neural networks with time-varying delay. Passivity analysis for neural networks of neutral type with Markovian jumping parameters and time delay in the leakage term has been presented in [18]. Robust exponential passive filtering for uncertain neutral-type neural networks with time-varying mixed delays via wirtinger-based integral inequality has been presented in [19] is Wirtinger-based integral inequality [20]. Recently, a new double integral inequality for time-delay system was proposed in [21], [22], which is less conservative. These motivate our research.

Motivated by above discussing, this paper investigates the passivity analysis for NTNNs with discrete and continuous distributed time-varying delays. Based on the constructed Lyapunov-Krasovskii functional, free-weighting matrix approach, and double integral inequality for estimating the derivative of the LyapunovKrasovskii functional, the delay-dependent passivity conditions are derived in terms of LMIs, which can be easily calculated by MATLAB LMIs control toolbox. Numerical example is provided to demonstrate the feasibility and effectiveness of the proposed criteria.

Notation: R^n is the n -dimensional Euclidean space; $R^{m\times n}$ denotes the set of $m\times n$ real matrices; I_n represents the n - dimensional identity matrix; $\lambda(\mathrm{A})$ denotes the set of all eigenvalues of A ; $\lambda_{\max}(\mathrm{A})=\max\{\mathrm{Re}\,\lambda;\lambda\in\lambda(\mathrm{A})\}$ $C([0,\mathrm{t}],\mathrm{R}^n)$ denotes the set of all R^n -valued continuous functions on $[0,t]$; $L_2([0,t],R^m)$ denotes the set of all the R^m -valued square integrable functions on $[0,\mathrm{t}]$; The notation $X\geq 0$ (respectively, $X>0$) means that X is positive semidefinite (respectively, positive definite); $\mathrm{diag}(\ldots)$ denotes a block diagonal matrix; Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation and Preliminaries

Consider the following NTNNs with time-varying discrete and distributed delays described by:

$$\begin{cases}\dot{x}(t)=-Ax(t)+Wg(x(t))+W_1g(x(t-\tau(t)))\\ \qquad +W_2\int_{t-k(t)}^{t}g(x(s))ds\\ \qquad +W_3\dot{x}(t-h(t))+u(t),\\ y(t)=g(x(t)),\\ x(t)=f(t),\quad t\in[-\tau_{\max},0],\\ \tau_{\max}=\max\{\tau_2,k_2,h_2\},\end{cases} \tag{1}$$

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where $x(t)=[x_1(t),x_2(t),\ldots,x_n(t)]\in R^n$ is the state of the neural, $A=\text{diag}(a_1,a_2,\ldots,a_n)>0$ represents the self-feedback term, W,W_1,W_2 and W_3 represents the connection weight matrices, $g(\cdot)=(g_1(\cdot),g_2(\cdot),\ldots,g_n(\cdot))^T$ represents the activation functions, $u(\text{t})$ and $y(\text{t})$ represents the input and output vectors, respectively; $\phi(\text{t})$ is an initial condition. The variables $\tau(t)$, $k(t)$ and $h(\text{t})$ represents the interval time-varying and time-varying delays with satisfy the following conditions:

$$\begin{cases} 0<\tau_1\le\tau(t)\le\tau_2, & \dot{\tau}(t)\le\tau_3, \\ 0\le h(t)\le h_2, & \dot{h}(t)\le h_3, \\ 0\le k(t)\le k_2, & \forall t\ge 0, \end{cases} \tag{2}$$

where known scalars $\tau_1,\tau_2,\tau_3,h_2,h_3$ and k_2.

The neural activation functions $g_k(\cdot),k=1,2,\ldots,n,$ satisfy $g_k(0)=0$ and for $s_1,s_2\in R,s_1\ne s_2$,

$$l_k^-\le\frac{g_k(s_1)-g_k(s_2)}{s_1-s_2}\le l_k^+, \tag{3}$$

where l_k^-,l_k^+ are known real scalars.

Definition 1 [13]: The system (1) is said to be passive if there exists a scalar γ such that for all $t_f\ge 0,$

$$2\int_0^{t_f}y^T(s)u(s)ds\ge-\gamma\int_0^{t_f}u^T(s)u(s)ds,$$

and for all solutions of (1) with $x(0)=0$.

Lemma 2 [21]: For a positive definite matrix $S>0,$ and any continuously differentiable function $x:[a,b]\to R^n$ the following inequality holds:

$$\int_a^b\dot{x}^T(s)S\dot{x}(s)ds\ge\frac{1}{b-a}\Pi_1^TS\Pi_1+\frac{3}{b-a}\Pi_2^TS\Pi_2+\frac{5}{b-a}\Pi_3^TS\Pi_3,$$

where

$$\Pi_1=x(b)-x(a),$$
$$\Pi_2=x(b)+x(a)-\frac{2}{b-a}\int_a^bx(s)ds,$$
$$\Pi_3=x(b)-x(a)+\frac{6}{b-a}\int_a^bx(s)ds-\frac{12}{(b-a)^2}\int_a^b\int_\theta^bx(s)ds\,d\theta.$$

Lemma 3 [22]: For a positive definite matrix $S>0,$ and any continuously differentiable function $x:[a,b]\to R^n$ the following inequality holds:

$$\int_a^b\int_\theta^b\dot{x}^T(s)S\dot{x}(s)ds\,d\theta\ge 2\Pi_4^TS\Pi_4+4\Pi_5^TS\Pi_5+6\Pi_6^TS\Pi_6,$$

where

$$\Pi_4=x(b)-\frac{1}{b-a}\int_a^bx(s)ds,$$
$$\Pi_5=x(b)+\frac{2}{b-a}\int_a^bx(s)ds-\frac{6}{(b-a)^2}\int_a^b\int_\theta^bx(s)ds\,d\theta,$$
$$\Pi_6=x(b)-\frac{3}{b-a}\int_a^bx(s)ds+\frac{24}{(b-a)^2}\int_a^b\int_\theta^bx(s)ds\,d\theta-\frac{60}{(b-a)^3}\int_a^b\int_\theta^b\int_s^bx(\lambda)d\lambda ds\,d\theta.$$

III. Main Results

For presentation convenience, in the following, we denote

$$L^+=\text{diag}(l_1^+,l_2^+,\ldots,l_n^+),$$
$$L^-=\text{diag}(l_1^-,l_2^-,\ldots,l_n^-),$$
$$\begin{aligned}\zeta(\text{t})=[&x^T(t),\ x^T(t-\tau(t)),\ x^T(t-\tau_1),\ x^T(t-\tau_2),\\ &g^T(x(t)),\ g^T(x(t-\tau(t))),\ g^T(x(t-\tau_1)),\\ &g^T(x(t-\tau_2)),\ \int_{t-\tau_1}^tx^T(s)ds,\ \int_{t-\tau_2}^tx^T(s)ds,\\ &\int_{t-\tau_1}^t\int_\theta^tx^T(s)ds\,d\theta,\ \int_{t-\tau_2}^t\int_\theta^tx^T(s)dsd\theta,\\ &\int_{t-\tau_1}^t\int_\theta^t\int_s^tx^T(\lambda)d\lambda dsd\theta,\\ &\int_{t-\tau_2}^t\int_\theta^t\int_s^tx^T(\lambda)d\lambda dsd\theta,\\ &\int_{t-k(t)}^tg^T(x(s))ds,\ \dot{x}^T(t-h(t)),\ u^T(t)]^T,\end{aligned}$$

and $e_i\in R^{n\times 17n}$ is defined as

$$e_i=[0_{n\times(i-1)n},\ I_n,\ 0_{n\times(17-i)n}]\ \text{ for } i=1,2,\ldots,17.$$

Theorem 1 For given scalars $\tau_1,\tau_2,\tau_3,h_2,h_3$ and k_2 the system (1) with (3) is passive for any delays satisfying (2), if there exist real positive matrices $P\in R^{7n\times 7n}$, $Q,\ S\in R^{n\times n},(i=1,2,3),$ real positive diagonal matrices $U_1,U_2,T_s,T_{ab}(\text{s}=1,2,3,4;\text{a}=1,2,3;b=2,3,4;a<b),$ with appropriate dimensions, and a scalar $\gamma>0$ such that the following LMIs holds:

$$\sum=\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7<0, \tag{4}$$

where

$$
\begin{cases}
\Omega_1 = \Pi_1^T P \Pi_2 + \Pi_2^T P \Pi_1 + (\Pi_3 + \Pi_4)^T \\
\qquad + \Pi_3 + \Pi_4, \\
\Omega_2 = 3e_1^T Q_1 e_1 - e_3^T Q_1 e_3 - e_4^T Q_1 e_4 + 2e_5^T Q_2 e_5 \\
\qquad - e_7^T Q_2 e_7 - e_8^T Q_2 e_8 + C_0^T Q_3 C_0 \\
\qquad - (1-\tau_3) e_6^T Q_1 e_6 - (1-h_3) e_{16}^T Q_3 e_{16}, \\
\Omega_3 = e_5^T k_2^2 S_1 e_5 - e_{15}^T S_1 e_{15}, \\
\Omega_4 = C_0^T (\tau_1^2 S_2 + \tau_2^2 S_2) C_0 - \Pi_5^T S_2 \Pi_5 \\
\qquad - 3\Pi_6^T S_2 \Pi_6 - 5\Pi_7^T S_2 \Pi_7 - \Pi_8^T S_2 \Pi_8 \\
\qquad - 3\Pi_9^T S_2 \Pi_9 - 5\Pi_{10}^T S_2 \Pi_{10}, \\
\Omega_5 = C_0^T (0.5\tau_1^2 S_3 + 0.5\tau_2^2 S_3) C_0 - 2\Pi_{11}^T S_3 \Pi_{11} \\
\qquad - 4\Pi_{12}^T S_3 \Pi_{12} - 6\Pi_{13}^T S_3 \Pi_{13} - 2\Pi_{14}^T S_3 \Pi_{14} \\
\qquad - 4\Pi_{15}^T S_3 \Pi_{15} - 6\Pi_{16}^T S_3 \Pi_{16}, \\
\Omega_6 = \sum_{s=1}^{4} (\Pi_{17}^T T_s \Pi_{18} + \Pi_{18}^T T_s \Pi_{17}) \\
\qquad + \sum_{a=1}^{3} \sum_{b=2,b>a}^{4} \Pi_{19}^T T_{ab} \Pi_{20} \\
\qquad + \sum_{a=1}^{3} \sum_{b=2,b>a}^{4} \Pi_{20}^T T_{ab} \Pi_{19}, \\
\Omega_7 = -\gamma e_{17}^T e_{17} - e_5^T e_{17} - e_{17}^T e_5,
\end{cases}
\tag{5}
$$

with

$$
\begin{aligned}
\Pi_1 &= [e_1^T, e_9^T, e_{10}^T, e_{11}^T, e_{12}^T, e_{13}^T, e_{14}^T]^T, \\
\Pi_2 &= [C_0^T, e_1^T - e_3^T, e_1^T - e_4^T, \tau_1 e_1^T - e_9^T, \tau_2 e_1^T - e_{10}^T, \\
&\quad 0.5\tau_1^2 e_1^T - e_{11}^T, 0.5\tau_2^2 e_1^T - e_{12}^T]^T, \\
\Pi_3 &= e_5^T (U_1 - U_2) C_0, \\
\Pi_4 &= e_1^T (L^+ U_2 - L^- U_1) C_0, \\
\Pi_5 &= e_1 - e_3, \\
\Pi_6 &= e_1 + e_3 - \frac{2}{\tau_1} e_9, \\
\Pi_7 &= e_1 - e_3 + \frac{6}{\tau_1} e_9 - \frac{12}{\tau_1^2} e_{11}, \\
\Pi_8 &= e_1 - e_4, \\
\Pi_9 &= e_1 + e_4 - \frac{2}{\tau_2} e_{10}, \\
\Pi_{10} &= e_1 - e_4 + \frac{6}{\tau_2} e_{10} - \frac{12}{\tau_2^2} e_{12}, \\
\Pi_{11} &= e_1 - \frac{1}{\tau_1} e_9, \\
\Pi_{12} &= e_1 + \frac{2}{\tau_1} e_9 - \frac{6}{\tau_1^2} e_{11}, \\
\Pi_{13} &= e_1 + e_2 - \frac{3}{\tau_1} e_9 + \frac{24}{\tau_1^2} e_{11} - \frac{60}{\tau_1^3} e_{13}, \\
\Pi_{14} &= e_1 - \frac{1}{\tau_2} e_{10}, \\
\Pi_{15} &= e_1 + \frac{2}{\tau_2} e_{10} - \frac{6}{\tau_2^2} e_{12}, \\
\Pi_{16} &= e_1 + e_2 - \frac{3}{\tau_2} e_{10} + \frac{24}{\tau_2^2} e_{12} - \frac{60}{\tau_2^3} e_{14}, \\
\Pi_{17} &= e_{s+4} - L^- e_s, \\
\Pi_{18} &= L^+ e_s - e_{s+4}, \\
\Pi_{19} &= (e_{a+4} - e_{b+4}) - L^+ (e_a - e_b), \\
\Pi_{20} &= L^+ (e_a - e_b) - (e_{a+4} - e_{b+4}), \\
C_0 &= A e_1 + W e_5 + W_1 e_6 + W_2 e_{15} + W_3 e_{16} + e_{17}.
\end{aligned}
$$

Proof: Consider a Lyapunov-Krasovskii functionai candidate:

$$V(x(t)) = \sum_{i=1}^{5} V_i(x(t)), \tag{6}$$

where

$$
\begin{aligned}
V_1(x(t)) &= \eta^T(t) P \eta(t) + 2\sum_{k=1}^{n} \rho_k \int_0^{x(t)} [g_k(s) - l_k^- s] ds \\
&\quad + 2\sum_{k=1}^{n} \sigma_k \int_0^{x(t)} [l_k^+ s - g_k(s)] ds, \\
V_2(x(t)) &= \sum_{i=1}^{2} \int_{t-\tau_i}^{t} [x^T(s) Q_1 x(s) \\
&\quad + g^T(x(s)) Q_2 g(x(s))] ds \\
&\quad + \int_{t-\tau(t)}^{t} x^T(s) Q_1 x(s) ds \\
&\quad + \int_{t-h(t)}^{t} \dot{x}^T(s) Q_3 \dot{x}(s) ds, \\
V_3(x(t)) &= k_2 \int_{t-k_2}^{t} \int_{\theta}^{t} g^T(x(s)) S_1 g(x(s)) ds\, d\theta, \\
V_4(x(t)) &= \sum_{i=1}^{2} \tau_i \int_{t-\tau_i}^{t} \int_{\theta}^{t} \dot{x}^T(s) S_2 \dot{x}(s) ds\, d\theta, \\
V_5(x(t)) &= \sum_{i=1}^{2} \int_{t-\tau_i}^{t} \int_{\theta}^{t} \int_{s}^{t} \dot{x}^T(\lambda) S_3 \dot{x}(\lambda) d\lambda ds d\theta,
\end{aligned}
$$

where

$$
\begin{aligned}
\eta(t) = [&x^T(t), \int_{t-\tau_1}^{t} x^T(s) ds, \int_{t-\tau_2}^{t} x^T(s) ds, \\
&\int_{t-\tau_1}^{t} \int_{\theta}^{t} x^T(s) ds d\theta, \int_{t-\tau_2}^{t} \int_{\theta}^{t} x^T(s) ds d\theta, \\
&\int_{t-\tau_1}^{t} \int_{\theta}^{t} \int_{s}^{t} x^T(\lambda) d\lambda ds d\theta, \\
&\int_{t-\tau_2}^{t} \int_{\theta}^{t} \int_{s}^{t} x^T(\lambda) d\lambda ds d\theta]^T,
\end{aligned}
$$

$U_1 = \mathrm{diag}\{\rho_1, \rho_2, \ldots, \rho_n\} \geq 0$, and $U_2 = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \geq 0$ are to be determined,

The time derivative of $V(x(t))$ can be computed as follows:

$$
\begin{aligned}
\dot{V}_1(x(t)) &= 2\dot{\eta}^T(t) P \eta(t) + 2\sum_{k=1}^{n} \{\rho_k \dot{x}(t)[g_k(x(t)) \\
&\quad - l_k^- x(t)] + \sigma_k \dot{x}(t)[l_k^+ x(t) - g_k(x(t))] \\
&\leq \varsigma^T(t) \Omega_1 \varsigma(t), \\
\dot{V}_2(x(t)) &\leq 3x^T(t) Q_1 x(t) + \dot{x}^T(t) Q_3 \dot{x}(t) \\
&\quad - x^T(t-\tau_1) Q_1 x(t-\tau_1) \\
&\quad - x^T(t-\tau_2) Q_1 x(t-\tau_2) \\
&\quad + 2g^T(x(t)) Q_2 g(x(t)) \\
&\quad - g^T(x(t-\tau_1)) Q_2 g(x(t-\tau_1)) \\
&\quad - g^T(x(t-\tau_2)) Q_2 g(x(t-\tau_2)) \\
&\quad - (1-\tau_3) x^T(t-\tau(t)) Q_1 x(t-\tau(t)) \\
&\quad - (1-h_3) \dot{x}^T(t-h(t)) Q_3 \dot{x}(t-h(t)) \\
&= \varsigma^T(t) \Omega_2 \varsigma(t),
\end{aligned}
$$

$$\dot{V}_3(x(t)) = k_2^2 g^T(x(t))S_1 g(x(t)) - k_2\int_{t-k_2}^{t} g^T(x(s))S_1 g(x(s))ds$$
$$\le k_2^2 g^T(x(t))S_1 g(x(t)) - k_2\int_{t-k(t)}^{t} g^T(x(s))S_1 g(x(s))ds$$
$$\le k_2^2 g^T(x(t))S_1 g(x(t)) - \int_{t-k(t)}^{t} g^T(x(s))dsS_1\int_{t-k(t)}^{t} g(x(s))ds$$
$$= \varsigma^T(t)\Omega_3\varsigma(t),$$

$$\dot{V}_4(x(t)) = \dot{x}^T(t)(\tau_1^2 S_2 + \tau_2^2 S_2)\dot{x}(t) - \tau_1\int_{t-\tau_1}^{t} \dot{x}^T(s)S_2\dot{x}(s)\,ds$$
$$\le \varsigma^T(t)\Omega_4\varsigma(t),$$

$$\dot{V}_5(x(t)) = 0.5\dot{x}^T(t)(\tau_1^2 S_3 + \tau_2^2 S_3)\dot{x}(t) - \int_{t-\tau_1}^{t}\int_{\theta}^{t}\dot{x}^T(s)S_3\dot{x}(s)\,dsd\theta - \int_{t-\tau_2}^{t}\int_{\theta}^{t}\dot{x}^T(s)S_3\dot{x}(s)\,dsd\theta$$
$$\le \varsigma^T(t)\Omega_5\varsigma(t),$$

where $\Omega_i, (i = 1,2,3,4,5)$ is defined in (5).

From (3), the nonlinear function $g_k(\mathrm{x}_k)$ satisfies

$$l_k^- \le \frac{g_k(x_k)}{x_k} \le l_k^+, \quad k = 1,2,\ldots,n, \ x_k \ne 0.$$

Thus, for any $t_k > 0, \ (k = 1,2,\ldots,\mathrm{n}),$ we have

$$2t_k[g_k^T(x(\theta)) - l_k^- x(\theta)][l_k^+ x(\theta) - g_k(x(\theta))] \ \ge 0,$$

which

$$2[g^T(x(\theta)) - x^T(\theta)L^-]^T T[L^+ x(\theta) - g(x(\theta))] \ge 0,$$

where $T = \mathrm{diag}\{t_1, t_2, \ldots, t_n\}.$

Let θ be $t, \ t-\tau(\mathrm{t}), \ t-\tau_1$ and $t-\tau_2,$ replace T with $T_s(s = 1,2,3,4),$ then, we have $(s = 1,2,3,4)$

$$2\varsigma^T(t)\Pi_{17}^T T_s \Pi_{18}\varsigma(t) \ \ge \ 0, \tag{7}$$

Another observation from (3), we have $l_k^- \le \frac{g_k(x(\theta_1)) - g_k(x(\theta_2))}{x(\theta_1) - x(\theta_2)} \le l_k^+, \ k = 1,2,\ldots,n.$ Thus, for any $t_k > 0, \ (k = 1,2,\ldots,\mathrm{n}),$ and $\Lambda = g_k(x(\theta_1)) - g_k(x(\theta_2)),$ we have

$$2t_k[\Lambda - l_k^-(x(\theta_1) - x(\theta_2))][l_k^+(x(\theta_1) - x(\theta_2)) - \Lambda] \ge 0,$$

which

$$2[\Lambda - L^-(x(\theta_1) - x(\theta_2))]^T \times T[L^+(x(\theta_1) - x(\theta_2)) - \Lambda] \ge 0,$$

where $\Lambda = \mathrm{col}\{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}.$

Let θ_1 and θ_2 take values in $t, t-\tau(\mathrm{t}), t-\tau_1$ and $t-\tau_2,$ and replace T with $T_{ab}(a = 1,2,3; \ b = 2,3,4; \ b > a),$ then, we have

$$2\varsigma^T(t)\Pi_{19}^T T_{ab}\Pi_{20}\varsigma(t) \ge 0, \tag{8}$$

where $a = 1,2,3, \ b = 2,3,4, \ b > a.$

From (7) and (8), it can be shown that

$$\varsigma^T(t)\Omega_6\varsigma(t) \ge 0, \tag{9}$$

where Ω_6 is defined in (5).

Therefore, we conclude that

$$\dot{V}(x(t)) - \gamma u^T(t)u(t) - 2y^T(t)u(t)$$
$$\le \sum_{i=1}^{5} V_i(x(t)) + \Omega_6 - \gamma u^T(t)u(t) - 2y^T(t)u(t)$$
$$= \varsigma^T(t) \sum \varsigma(t), \tag{10}$$

where $\sum$ is defined in (4). If we have $\sum < 0,$ then

$$\dot{V}(x(t)) - \gamma u^T(t)u(t) - 2y^T(t)u(t) \le 0, \tag{11}$$

for any $\varsigma(t) \ne 0.$ Since $V(x(0)) = 0$ under zero initial condition, let $x(t) = 0$ for $t \in [\tau_{\max}, 0],$ after integrating (11) with respect to t over the time period from 0 to t_f we get

$$2\int_0^{t_f} y^T(s)u(s)ds$$
$$\ge \ V(x(t_f)) - V(x(0)) - \gamma\int_0^{t_f} u^T(s)u(s)ds$$
$$\ge \ -\gamma\int_0^{t_f} u^T(s)u(s)ds.$$

Thus, the NTNNs (1) with (3) is passive in the sense of Definition 1. This completes the proof.

IV. Numerical Example

In this section, we present example to illustrate the effectiveness and the reduced conservatism of our result. Consider the NTNNs (1) with the following parameters:

$$A = \begin{bmatrix} 8.4 & 0 \\ 0 & 9 \end{bmatrix}, \qquad W = \begin{bmatrix} -0.21 & -0.19 \\ -0.24 & 0.1 \end{bmatrix},$$
$$W_1 = \begin{bmatrix} -0.09 & -0.2 \\ 0.2 & 0.1 \end{bmatrix}, W_2 = \begin{bmatrix} -0.52 & 0 \\ 0.2 & -0.09 \end{bmatrix},$$
$$W_3 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}.$$

The activation functions are assumed to be

$$g_i(x_i(t)) = 0.5(|x_i+1|-|x_i-1|), i=1,2.$$

It is easy to see:

$$L^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, L^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For $0.2 \le \tau(t) \le 3,\ h(t) \le 0.5,\ k(t) \le 2,\ h_3 = 0.5,$ $\tau_3 = 0.5$ and $\gamma = 0.34$ by using MATLAB LMIs control toolbox and by solving the LMIs in Theorem 1, in our paper we obtain the feasible solutions:

$$P = \left[\begin{array}{cccccccccccccc}
1.0087 & 0.1196 & -0.0038 & -0.0013 & 0.0000 & 0.0000 & -0.0286 & -0.0035 & 0.0000 & 0.0000 & -0.0060 & 0.0030 & 0.0000 & 0.0000 \\
0.1196 & 0.7258 & -0.0067 & -0.0088 & 0.0000 & 0.0000 & -0.0289 & -0.0472 & 0.0000 & 0.0000 & 0.0007 & 0.0050 & 0.0000 & 0.0000 \\
-0.0038 & -0.0067 & 0.0128 & 0.0084 & -0.0016 & -0.0015 & 0.0098 & -0.0263 & -0.0022 & -0.0011 & 0.0174 & -0.0033 & -0.0001 & 0.0000 \\
-0.0013 & -0.0088 & 0.0084 & 0.0429 & -0.0017 & -0.0056 & 0.0892 & 0.0502 & -0.0013 & -0.0050 & 0.0289 & 0.0096 & 0.0000 & -0.0002 \\
0.0000 & 0.0000 & -0.0016 & -0.0017 & 0.0012 & 0.0011 & 0.0007 & 0.0001 & 0.0002 & 0.0003 & 0.0005 & 0.0002 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -0.0015 & -0.0056 & 0.0011 & 0.0040 & 0.0000 & 0.0007 & 0.0003 & 0.0010 & 0.0001 & 0.0006 & 0.0000 & 0.0000 \\
-0.0286 & -0.0289 & 0.0098 & 0.0892 & 0.0007 & 0.0000 & 1.4958 & 0.7472 & -0.0009 & -0.0003 & -0.0140 & -0.1875 & -0.0001 & 0.0000 \\
-0.0035 & -0.0472 & -0.0263 & 0.0502 & 0.0001 & 0.0007 & 0.7472 & 3.0171 & -0.0004 & -0.0020 & -0.0364 & -0.3777 & 0.0000 & -0.0001 \\
0.0000 & 0.0000 & -0.0022 & -0.0013 & 0.0002 & 0.0003 & -0.0009 & -0.0004 & 0.0006 & 0.0004 & 0.0001 & 0.0002 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -0.0011 & -0.0050 & 0.0003 & 0.0010 & -0.0003 & -0.0020 & 0.0004 & 0.0014 & 0.0002 & 0.0004 & 0.0000 & 0.0000 \\
-0.0060 & 0.0007 & 0.0174 & 0.0289 & 0.0005 & 0.0001 & -0.0140 & -0.0364 & 0.0001 & 0.0002 & 5.1380 & 0.2050 & 0.0000 & 0.0000 \\
0.0030 & 0.0050 & -0.0033 & 0.0096 & 0.0002 & 0.0006 & -0.1875 & -0.3777 & 0.0002 & 0.0004 & 0.2050 & 5.7050 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -0.0001 & 0.0000 & 0.0000 & 0.0000 & -0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & -0.0002 & 0.0000 & 0.0000 & 0.0000 & -0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000
\end{array}\right],$$

$$Q_1 = \begin{bmatrix} 0.2383 & 0.3599 \\ 0.3599 & 0.7875 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.0019 & 0.0023 \\ 0.0023 & 0.0093 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0.3563 & 0.0107 \\ 0.0107 & 0.2513 \end{bmatrix}, S_1 = \begin{bmatrix} 2.2291 & 0.0099 \\ 0.0099 & 0.2674 \end{bmatrix},$$

$$S_2 = 10^{-5} \times \begin{bmatrix} 2.6041 & 1.1983 \\ 1.1983 & 5.4488 \end{bmatrix},$$

$$S_3 = 10^{-8} \times \begin{bmatrix} 5.4115 & -0.0914 \\ -0.0914 & 5.2782 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 2.0621 & 0 \\ 0 & 1.5664 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 3.9452 & 0 \\ 0 & 3.2423 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 23.2649 & 0 \\ 0 & 16.9973 \end{bmatrix}, T_2 = \begin{bmatrix} 0.0792 & 0 \\ 0 & 0.1857 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0.0562 & 0 \\ 0 & 0.1528 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0.0025 & 0 \\ 0 & 0.0141 \end{bmatrix},$$

$$T_5 = \begin{bmatrix} 1.2748 & 0 \\ 0 & 1.3177 \end{bmatrix}, \quad T_6 = \begin{bmatrix} 1.2244 & 0 \\ 0 & 1.3813 \end{bmatrix},$$

$$T_7 = \begin{bmatrix} 1.2681 & 0 \\ 0 & 1.2956 \end{bmatrix}, \quad T_8 = \begin{bmatrix} 1.3421 & 0 \\ 0 & 1.2576 \end{bmatrix},$$

$$T_9 = \begin{bmatrix} 1.2748 & 0 \\ 0 & 1.3177 \end{bmatrix}, \quad T_{10} = \begin{bmatrix} 1.0461 & 0 \\ 0 & 1.0442 \end{bmatrix}.$$

V. Conclusion

In this paper, the passivity analysis for NTNNs with discrete and distributed time-varying delays has been studied. By employing the LKFs method, double integral inequality was developed to guarantee the passivity performance of NTNNs. A new passivity analysis criterion has been given in terms of LMIs, which is dependent on time-varying delays. Finally, a numerical example has been presented which illustrate the effectiveness and usefulness of the proposed method.

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