MacWilliams Identities of Linear Codes over a Matrix Ring with Respect to Rosenbloom-Tsfasman Metric

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***Abstract*—The definitions of ρ complete weight enumerator and exact complete ρ weight enumerator over a matrix ring are given, and the MacWilliams identities with respect to RT metric for the two weight enumerators of linear codes over the matrix ring are obtained, respectively. Finally, we give an example to illustrate our obtained results.**

***Index Tems*—RT metric, weight enumerator, MacWilliams identity, dual code.**

I. Introduction

Error-correcting codes has been widely used in the field of computer and communication, and the MacWilliams identity is one of the important results in error-correcting codes. There has been a recent growth of interest in linear codes with respect to a newly defined non-Hamming metric grown as the Rosenbloom-Tsfassman metric (RT, or ρ in short) [1]. Siap have done a lot of contribution to the MacWilliams identities with respect to ρ complete weight enumerators over alphabets $M_{n\times s}(F_q)$, $F_q[u]/(u^r-a)$ and $M_{n\times s}(R)$ in [2]-[4]. MacWilliams identities for ρ complete weight enumerators of codes over Z_4 are given in [5], and the case over $M_{n\times s}(Z_4)$ was considered in [6]. Recently, MacWilliams identities over non-chain rings were considered in different literatures [7]-[10], and the Lee ρ weight enumerator and exact complete ρ weight enumerator of linear codes over $F_p+vF_p+v^2F_p$ were obtained in [11]. Note that codes over Z_4+vZ_4 with RT metric were studied in [12].

Motivated by the work listed above, we continue to consider MacWilliams identities of linear codes over $M_{n\times s}(Z_{l^m}+vZ_{l^m})$. This paper is devoted to the determination of the MacWilliams identities of linear codes with respect to the Lee complete ρ weight enumerator and the exact complete ρ weight enumerator of linear codes over $M_{n\times s}(R)$, where $R=Z_{l^m}+vZ_{l^m}$ ($v^2=v$) and using this we derive the MacWilliams identities for linear codes over *R*.

II. Preliminaries

For convenience, we let R be the commutative ring $R=Z_{l^m}+vZ_{l^m}=\{a+bv\mid a,b\in Z_{l^m}\}$, where $v^2=v$, l is prime. Note that ring Z_4+vZ_4 in [1] and ring F_2+vF_2 in [12] are special cases of $Z_{l^m}+vZ_{l^m}$. Let $M_{n\times s}(R)$ be the set of all $n\times s$ matrices over *R*. Let $\mathbf{p}=(p_0,p_1,\cdots,p_{s-1})\in M_{1\times s}(R)$, we define the ρ weight of $\mathbf{p}$ by

$$w_H(\mathbf{p})=\begin{cases}\max\{i: p_i\neq 0\}+1, & \mathbf{p}\neq\mathbf{0},\\ 0, & \mathbf{p}=\mathbf{0}.\end{cases}\qquad(1)$$

and define the RT distance between $\mathbf{p}$ and $\mathbf{q}$ by $\rho(\mathbf{p},\mathbf{q})=w_H(\mathbf{p}-\mathbf{q})$, where $\mathbf{p},\ \mathbf{q}\in M_{1\times s}(R)$. Let $P=(P_1,P_2,\cdots,P_n)^{\mathrm{T}}\in M_{n\times s}(R)$, where $P_i=(p_{i,0},p_{i,1},\cdots,p_{i,s-1})\in M_{1\times s}(R)$, $1\le i\le n$, the ρ weight is then extended as $w_H(P)=\sum_{i=1}^n w_H(P_i)$, the RT distance between P and Q is given as $\rho(P,Q)=w_H(P-Q)$, where $P,\ Q\in M_{n\times s}(R)$. Note that for $s=1$ the metric is the usual Hamming metric.

A linear code over $M_{n\times s}(R)$ is an R-submodule of $M_{n\times s}(R)$. Let $C\subseteq M_{n\times s}$ be a linear code, the set $w_r(C)=|\{P\in C|w_H(P)=r\}|$, where $0\le r\le ns$, is called the weight spectrum of C and the ρ weight enumerator of C is defined by

$$W_C(z)=\sum_{r=0}^{ns}w_r(C)z^r=\sum_{P\in C}z^{w_H(P)}.\qquad(2)$$

Let $\mathbf{p}=(p_0,p_1,\cdots,p_{s-1})$, $\mathbf{q}=(q_0,q_1,\cdots,q_{s-1})\in M_{1\times s}(R)$ the inner product of $\mathbf{p}$ and $\mathbf{q}$ is defined by $<\mathbf{p},\mathbf{q}>=\sum_{i=0}^{s-1}p_iq_{s-1-i}$ and this is extended to the inner product of $P=(P_1,P_2,\cdots,P_n)^{\mathrm{T}}$, $Q=(Q_1,Q_2,\cdots,Q_n)^{\mathrm{T}}\in M_{n\times s}(R)$ as $<P,Q>=\sum_{i=1}^n<P_i,Q_i>$.

Definition 2.1 The dual $C^{\perp}$ of a linear code C over $M_{n\times s}(R)$ is defined by $C^{\perp}=\{Q\in M_{n\times s}(R)|<P,Q>=0$, $\forall P\in C\}$. It is clear that $C^{\perp}$ is also a linear code over $M_{n\times s}(R)$.

The ring of $n\times s$ matrices over R can be identified with the ring of $n\times 1$ matrices having polynomial entries. We identify the set of all polynomials of degree at most s − 1 over with $R[x]/(x^s)$. Define map: φ

$$M_{n\times s}(R)\ \to M_{n\times 1}(R[x]/(x^s))$$
$$P\ \ \to(p_{1,0}+p_{1,1}x+\cdots+p_{1,s-1}x^{s-1},\cdots,p_{n,0}+p_{n,1}x+\cdots+p_{n,s-1}x^{s-1})^{\mathrm{T}},$$

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where $P=(P_1,P_2,\cdots,P_n)^{\mathrm{T}}$ and $P_i=(p_{i,0},p_{i,1},\cdots,p_{i,s-1})$, then the map defined above is an R-module isomorphism. It is easy to check that the ρ weight w_H of a polynomial $p(x)\in R[x]/(x^s)$ is $\deg(p(x))+1$.

Let $p(x)=p_0+p_1x+\cdots+p_{s-1}x^{s-1}\in R[x]/(x^s)$, the ℓ th coefficient of $p(x)$ is defined by $c_\ell(p(x))=p_\ell$, $0\le\ell\le s-1$. Then the inner product $<p(x),q(x)>$ becomes $c_{s-1}(p(x)q(x))$. Similarly, suppose $P(x)=(P_1(x),P_2(x),\cdots,P_n(x))^{\mathrm{T}}$ and $Q(x)=(Q_1(x),Q_2(x),\cdots,Q_n(x))^{\mathrm{T}}\in M_{n\times1}(R[x]/(x^s))$, the inner product of $P(x)$ and $Q(x)$ defined above in terms of polynomials becomes $<P(x),Q(x)>=\sum_{i=1}^{n}c_{s-1}(P_i(x)Q_i(x))$. Furthermore, for $0\in Z_{l^m}$, the Hamming weight $w(0)$ of zero element is 0, otherwise 1.

Definition 2.2 Let $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots y_{n,0},\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\le i\le n$, $0\le j\le s-1$. Define ρ complete weight enumerator of a code C over $M_{n\times s}(R)$ as

$$W_C(Y_{ns})=\sum_{P\in C}y_{1,0}^{w(p_{1,0})}\cdots y_{1,s-1}^{w(p_{1,s-1})}\cdots y_{n,0}^{w(p_{n,0})}\cdots y_{n,s-1}^{w(p_{n,s-1})}. \quad (3)$$

When $n=1$, $s=r$, we have $Y=(y_1,y_2,\cdots,y_r)$, we get the ρ complete weight enumerator of a code C over R as $W_C(Y)=\sum_{\mathbf{p}\in C}y_1^{w(p_0)}y_2^{w(p_1)}\cdots y_r^{w(p_{r-1})}$, where $\mathbf{p}=(p_0,p_1,\cdots,p_{r-1})\in C$.

III. The Lee Complete ρ Weight Enumerator

In this section, we will give a MacWilliams identity with respect to Lee complete ρ weight enumerator over $M_{n\times s}(R)$.

Firstly, we give a Gray map between R (Lee weight) and $Z_{l^m}^2$ (Hamming weight) as follows:

Definition 3.1 Define Gray map τ : $R\to Z_{l^m}^2$, $\tau(a+bv)=(a,a+b)$, $\forall a+bv\in R$, where $a,b\in Z_{l^m}$.

According to Definition 3.1, it is easy to verify that the Lee weight of C is the Hamming weight of its Gray image τ. Moreover, the Gray map τ is an isometry from (R^n, Lee distance) to ($Z_{l^m}^{2n}$, Hamming distance).

Definition 3.2 For $\forall\alpha\in R$, we define its Lee weight as

$$w_L(\alpha)=\begin{cases}0, & if\ \alpha=0,\\ 1, & if\ \alpha=av, a+(l^m-a)v,\\ 2, & if\ \alpha=a, a+bv,\end{cases} \quad (4)$$

where $a,b\in Z_{l^m}\setminus\{0\}$ and $a+b\ne0\ (\mathrm{mod}\ l^m)$.

Definition 3.3 Let $Y=(y_1,y_2,\cdots,y_r)$, $\mathbf{p}=(p_0,p_1,\cdots,p_{r-1})\in R^r$, the Lee complete ρ weight enumerator of a code C is defined as

$$W_C(Y)=\sum_{\mathbf{p}\in C}y_1^{w_L(p_0)}y_2^{w_L(p_1)}\cdots y_r^{w_L(p_{r-1})}.$$

Moreover, let $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots y_{n,0},\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, $1\le i\le n$, $0\le j\le s-1$. we define the Lee complete ρ weight enumerator of a code C over $M_{n\times s}(R)$ as

$$W_C(Y_{ns})=\sum_{P\in C}y_{1,0}^{w_L(p_{1,0})}\cdots y_{1,s-1}^{w_L(p_{1,s-1})}\cdots y_{n,0}^{w_L(p_{n,0})}\cdots y_{n,s-1}^{w_L(p_{n,s-1})}. \quad (5)$$

In (5), we let $n=r$, $s=1$ and arrange the subscripts properly, then we obtain a new Lee weight enumerator, that is: $W_C(Y)^*=\sum_{\mathbf{p}\in C}y_1^{w_L(p_0)}y_2^{w_L(p_1)}\cdots y_r^{w_L(p_{r-1})}$, where $\mathbf{p}=(p_0,\ p_1,\cdots,p_{r-1})$ and $Y=(y_1,y_2,\cdots,y_r)$, which is called the Lee weight enumerator of a code C over R; when $n=1$, $s=r$, by properly interchanging the subscripts, we get (4).

Definition 3.4 The map $\chi:\ R\to C^*$, $\chi(a+bv)=\xi^b$ $\forall a+bv\in R$, where $\xi=e^{2\pi i/l^m}$, then χ is a nontrivial character of C.

Similar to the proof of Lemmas 2 and 2.3 in [9] and [7], respectively, we can obtain the following Lemmas,

Lemma 3.5 Let C be a linear code over $M_{n\times s}(R)$, $P(x)$, $Q(x)\in M_{n\times1}(R[x]/(x^s))$, then we have

$$\sum_{P(x)\in C}\chi(<P(x),Q(x)>)=\begin{cases}0, & if\ Q(x)\notin C^\perp,\\ |C|, & if\ Q(x)\in C^\perp.\end{cases} \quad (6)$$

Lemma 3.6 Let $f:M_{n\times1}(R[x]/(x^s))\to C[y_{1,0},\cdots y_{1,s-1},\cdots,y_{n,0},\cdots y_{n,s-1}]$, and χ be defined above, then

$$\sum_{Q(x)\in C^\perp}f(Q(x))=\frac{1}{|C|}\sum_{P(x)\in C}\hat{f}(P(x)), \quad (7)$$

where $\hat{f}(P(x))=\sum_{Q(x)\in M_{n\times1}(R[x]/(x^s))}\chi(<P(x),Q(x)>)f(Q(x))$.

The following lemma will play an important role in obtaining the main result.

Lemma 3.7 Let β be a fixed element of R, then we have

$$\sum_{\alpha\in R}\chi(\beta\alpha)y^{w_L(\alpha)}=[1+(l^m-1)y]^{2-w_L(\beta)}(1-y)^{w_L(\beta)}.$$

Proof. We only prove the case when $l=2$, $m=2$, and the proof is similar when l is odd prime. In light of Definitions 3.2 and 3.4, we have

$$\sum_{\alpha\in R}\chi(\beta\alpha)y^{w_L(\alpha)}=\begin{cases}(1+3y)^2, & \beta=0,\\ (1+3y)(1-y), & \beta=v,2v,3v,1+3v,2+2v,3+v,\\ (1-y)^2, & \beta=1,2,3,1+v,1+2v,2+v,2+3v,3+2v,3+3v,\end{cases}$$
$$=(1+3y)^{2-w_L(\beta)}(1-y)^{w_L(\beta)}.$$

Now we will give the first main result, that is, we give a MacWilliams identity of linear codes over $M_{n\times s}(R)$ with respect to Lee complete ρ weight enumerator.

Theorem 3.8 Let C be a linear code over $M_{n\times s}(R)$, then

$$\sum_{Q(x)\in C^{\perp}} y_{1,0}^{w_L(q_{1,0})}\cdots y_{1,s-1}^{w_L(q_{1,s-1})}\cdots y_{n,0}^{w_L(q_{n,0})}\cdots y_{n,s-1}^{w_L(q_{n,s-1})}$$

$$=\frac{1}{|C|}\sum_{P(x)\in C}\prod_{i=1}^{n}\prod_{j=0}^{s-1}\left[1+(l^m-1)y_{i,j}\right]^2\times\prod_{k=1}^{n}\prod_{t=0}^{s-1}\left(\frac{1-y_{k,t}}{1+(l^m-1)y_{k,t}}\right)^{w_L(p_{k,s-1-t})}.$$

Proof. Suppose $f(Q(x))=f((Q_1(x),Q_2(x),\cdots,Q_n(x))^{\mathrm{T}})=\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{w_L(q_{i,j})}$, by applying Lemmas 3.5 and 3.7, then

$$\begin{aligned}\hat{f}(P(x))&=\sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))}\chi(<P(x),Q(x)>)f(Q(x))\\
&=\sum_{q_{1,0}\in R}\chi(<P_1(x),q_{1,0}>)y_{1,0}^{w_L(q_{1,0})}\cdots\\
&\quad\sum_{q_{1,s-1}\in R}\chi(<P_1(x),q_{1,s-1}x^{s-1}>)y_{1,s-1}^{w_L(q_{1,s-1})}\cdots\\
&\quad\sum_{q_{n,0}\in R}\chi(<P_n(x),q_{n,0}>)y_{n,0}^{w_L(q_{n,0})}\cdots\\
&\quad\sum_{q_{n,s-1}\in R}\chi(<P_n(x),q_{n,s-1}x^{s-1}>)y_{n,s-1}^{w_L(q_{n,s-1})}\\
&=[1+(l^m-1)y_{1,0}]^{2-w_L(p_{1,s-1})}(1-y_{1,0})^{w_L(p_{1,s-1})}\cdots\\
&\quad[1+(l^m-1)y_{1,s-1}]^{2-w_L(p_{1,0})}(1-y_{1,s-1})^{w_L(p_{1,0})}\cdots\\
&\quad[1+(l^m-1)y_{n,0}]^{2-w_L(p_{n,s-1})}(1-y_{n,0})^{w_L(p_{n,s-1})}\cdots\\
&\quad[1+(l^m-1)y_{n,s-1}]^{2-w_L(p_{n,0})}(1-y_{n,s-1})^{w_L(p_{n,0})}\\
&=\prod_{i=1}^{n}\prod_{j=0}^{s-1}\left[1+(l^m-1)y_{i,j}\right]^{2-w_L(p_{i,s-1-j})}\left(1-y_{i,j}\right)^{w_L(p_{i,s-1-j})}\\
&=\prod_{i=1}^{n}\prod_{j=0}^{s-1}\left[1+(l^m-1)y_{i,j}\right]^2\times\prod_{k=1}^{n}\prod_{t=0}^{s-1}\left(\frac{1-y_{k,t}}{1+(l^m-1)y_{k,t}}\right)^{w_L(p_{k,s-1-t})}.\end{aligned}$$

the result then follows by applying Lemma 3.6.

The following assertion is the special cases of Theorem 3.8, so we omit its proof.

Corollary 3.9 Let C be a linear code over R, $q(x)=q_0+q_1x+\cdots+q_{r-1}x^{r-1}$, $p(x)=p_0+p_1x+\cdots+p_{r-1}x^{r-1}\in R[x]/(x^r)$, then we have

1) In Theorem 3.8, let n=1, s=r, by properly interchang-ing subscripts, then

$$\sum_{q(x)\in C^{\perp}} y_1^{w_L(q_0)}y_2^{w_L(q_1)}\cdots y_r^{w_L(q_{r-1})}=\frac{1}{|C|}\sum_{p(x)\in C}\prod_{i=1}^{r}\left[1+(l^m-1)y_i\right]^2\prod_{k=1}^{r}\left(\frac{1-y_k}{1+(l^m-1)y_k}\right)^{w_L(p_{r-k})},$$

which is called the MacWilliams identity with respect to Lee complete ρ weight enumerator of the linear code C over R.

2) In Theorem 3.8, let s=1, n=r, by properly interchange-ing subscripts, then

$$\sum_{q(x)\in C^{\perp}} y_1^{w_L(q_0)}y_2^{w_L(q_1)}\cdots y_r^{w_L(q_{r-1})}=\frac{1}{|C|}\sum_{p(x)\in C}\prod_{i=1}^{r}[1+(l^m-1)y_i]^2\prod_{k=1}^{r}\left(\frac{1-y_k}{1+(l^m-1)y_k}\right)^{w_L(p_{k-1})},$$

which is called the MacWilliams identity with respect to Lee weight enumerator of the linear code C over R.

IV. The Exact Complete ρ Weight Enumerator

In this section, we will give a MacWilliams identity with respect to exact complete ρ weight enumerator over $M_{n\times s}(R)$.

Definition 4.1 For $\forall a+bv\in R$, the exact weight of $a+bv$ is defined as $w_e(a+bv)=a+bl^m$, where $a,b\in Z_{l^m}$.

Definition 4.2 Let $Y_{ns}=(y_{1,0},\cdots y_{1,s-1},\cdots y_{n,0}\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\le i\le n$, $0\le j\le s-1$, we define the exact complete ρ weight enumerator as

$$E_C(Y_{ns})=\sum_{P\in C}y_{1,0}^{w_e(p_{1,0})}\cdots y_{1,s-1}^{w_e(p_{1,s-1})}\cdots y_{n,0}^{w_e(p_{n,0})}\cdots y_{n,s-1}^{w_e(p_{n,s-1})},\quad(8)$$

especially, in (8), we let $n=1$, $s=r$, and by properly inter-changing subscripts, we can obtain the exact complete ρ weight enumerator of a code C over R:

$$E_C(Y)=\sum_{\mathbf{p}\in C}y_1^{w_e(p_0)}y_2^{w_e(p_1)}\cdots y_r^{w_e(p_{r-1})},$$

where $Y=(y_1,y_2,\cdots,y_r)$ and $\mathbf{p}=(p_0,p_1,\cdots,p_{r-1})$.

Now, let us prove the following lemma, which plays an important role in the main result.

Lemma 4.3 Let β be a fixed element of R, then we have

$$\sum_{\alpha\in R}\chi(\beta\alpha)y^{w_e(\alpha)}=\prod_{k=1}^{m}(1+\sum_{i=1}^{l-1}\chi(il^{k-1}\beta)y^{il^{k-1}})(1+\sum_{j=1}^{l-1}\chi(jl^{k-1}\beta v)y^{jl^{m+k-1}}).$$

Proof. We only prove the case when $l=3$, $m=1$, and the proof is similar when l is prime except the case when $l=3$. In light of Definitions 3.4 and 4.1, we have

$$\sum_{\alpha\in R}\chi(\beta\alpha)y^{w_e(\alpha)}=\begin{cases}(1+y+y^2)(1+y^3+y^6), & \beta=0,\\ (1+y+y^2)(1+\xi y^3+\xi^2y^6), & \beta=1,\\ (1+y+y^2)(1+\xi^2y^3+\xi y^6), & \beta=2,\\ (1+\xi y+\xi^2y^2)(1+\xi y^3+\xi^2y^6), & \beta=v,\\ (1+\xi y+\xi^2y^2)(1+\xi^2y^3+\xi y^6), & \beta=1+v,\\ (1+\xi y+\xi^2y^2)(1+y^3+y^6), & \beta=2+v,\\ (1+\xi^2y+\xi y^2)(1+\xi^2y^3+\xi y^6), & \beta=2v,\\ (1+\xi^2y+\xi y^2)(1+y^3+y^6), & \beta=1+2v,\\ (1+\xi^2y+\xi y^2)(1+\xi y^3+\xi^2y^6), & \beta=2+2v,\end{cases}$$

$$=(1+\chi(\beta)y+\chi(2\beta)y^2)(1+\chi(\beta v)y^3+\chi(2\beta v)y^6).$$

Theorem 4.4 Let C be a linear code over $M_{n\times s}(R)$, then

$$\sum_{Q(x)\in C^{\perp}}y_{1,0}^{w_e(q_{1,0})}\cdots y_{1,s-1}^{w_e(q_{1,s-1})}\cdots y_{n,0}^{w_e(q_{n,0})}\cdots y_{n,s-1}^{w_e(q_{n,s-1})}$$

$$=\frac{1}{|C|}\sum_{P(x)\in C}\prod_{k=1}^{m}\prod_{t=1}^{n}\prod_{j=0}^{s-1}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{t,s-1-j})y_{t,j}^{ip^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{t,s-1-j}v)y_{t,j}^{i'l^{m+k-1}}\right].$$

Proof. Suppose $f(Q(x))=f((Q_1(x),Q_2(x),\cdots,Q_n(x))^{\mathrm{T}})=\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{w_e(q_{i,j})}$, by applying Lemmas 3.5 and 4.3, then

$$\hat{f}(P(x)) = \sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))} \chi(<P(x),Q(x)>) f(Q(x))$$

$$= \sum_{q_{1,0}\in R} \chi(<P_1(x),q_{1,0}>) y_{1,0}^{w_e(q_{1,0})} \cdots \sum_{q_{1,s-1}\in R} \chi(<P_1(x),q_{1,s-1}x^{s-1}>) y_{1,s-1}^{w_e(q_{1,s-1})} \cdots$$

$$\sum_{q_{n,0}\in R} \chi(<P_n(x),q_{n,0}>) y_{n,0}^{w_e(q_{n,0})} \cdots \sum_{q_{n,s-1}\in R} \chi(<P_n(x),q_{n,s-1}x^{s-1}>) y_{n,s-1}^{w_e(q_{n,s-1})}$$

$$= \prod_{k=1}^{m}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{1,s-1}) y_{1,0}^{il^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{1,s-1}v) y_{1,0}^{i'l^{m+k-1}}\right]\cdots$$

$$\prod_{k=1}^{m}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{1,0}) y_{1,s-1}^{il^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{1,0}v) y_{1,s-1}^{i'l^{m+k-1}}\right]\cdots$$

$$\prod_{k=1}^{m}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{n,s-1}) y_{n,0}^{il^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{n,s-1}v) y_{n,0}^{i'l^{m+k-1}}\right]\cdots$$

$$\prod_{k=1}^{m}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{n,0}) y_{n,s-1}^{il^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{n,0}v) y_{n,s-1}^{i'l^{m+k-1}}\right]$$

$$= \prod_{k=1}^{m}\prod_{t=1}^{n}\prod_{j=0}^{s-1}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{t,s-1-j}) y_{t,j}^{il^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{t,s-1-j}v) y_{t,j}^{i'l^{m+k-1}}\right].$$

the result then follows by applying Lemma 4.3.

The following assertion is the special cases of Theorem 4.4, so we omit its proof.

Corollary 4.5 Let C be a linear code over R, $q(x)=q_0 +q_1x+\cdots+q_{r-1}x^{r-1}$, $p(x)=p_0+p_1x+\cdots+p_{r-1}x^{r-1}\in R[x]/(x^r)$, then we have

1) Let n=1, s=r in Theorem 4.4, by properly interchang-ing subscripts, then

$$\sum_{q(x)\in C^{\perp}} y_1^{w_e(q_0)} y_2^{w_e(q_1)} \cdots y_r^{w_e(q_{r-1})}$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}\prod_{k=1}^{m}\prod_{j=1}^{r}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{r-j}) y_j^{il^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{r-j}v) y_j^{i'l^{m+k-1}}\right],$$

which is called the MacWilliams identity with respect to exact complete ρ weight enumerator of the linear code C over R.

2) In Theorem 4.4, let s=1, n=r, by properly interchange-ing subscripts, then

$$\sum_{q(x)\in C^{\perp}} y_1^{w_e(q_0)} y_2^{w_e(q_1)} \cdots y_r^{w_e(q_{r-1})}$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}\prod_{k=1}^{m}\prod_{j=1}^{r}\left[1+\sum_{i=1}^{l-1}\chi(il^{k-1}p_{j-1}) y_j^{il^{k-1}}\right]\left[1+\sum_{i'=1}^{l-1}\chi(i'l^{k-1}p_{j-1}v) y_j^{i'l^{m+k-1}}\right],$$

which is called the MacWilliams identity with respect to exact weight enumerator of the linear code C over R.

V. Illustration Examples

In the previous Sections, MacWilliams identities with respect to Lee complete ρ weight enumerator and exact complete ρ weight enumerator over ring $M_{n\times s}(R)$ are obtained. In this section, we mainly present an example to illustrate the application of main results.

Let $C=\{(0,0),(v,v)\}$ be a linear code of length 2 over ring Z_2+vZ_2, we have the generator matrix of $C^{\perp}$ is $G=\begin{pmatrix} 1 & 1 \\ 1+v & 0 \end{pmatrix}$, that is, $C^{\perp}$={(0, 0), (1, 1), (v, v), (1+v, 1+v),(1+v, 0), (0,1+v), (v,1), (1,v)}. According to Definition 3.3, we have $Lee_C(Y)=1+y_1y_2$, and then by Corollary 3.9,

$$Lee_{C^{\perp}}(Y) = \frac{1}{|C|}\sum_{p(x)\in C}\prod_{i=1}^{2}[1+(2-1)y_i]^2\prod_{k=1}^{2}\left(\frac{1-y_k}{1+(2-1)y_k}\right)^{w_L(p_{2-k})}$$

$$= \frac{1}{2}(1+y_1)^2(1+y_2)^2\left(1+\left(\frac{1-y_1}{1+y_1}\right)\left(\frac{1-y_2}{1+y_2}\right)\right)$$

$$= 1+y_1+y_2+2y_1y_2+y_1^2y_2+y_1y_2^2+y_1^2y_2^2.$$

On the other hand, according to Definition 3.3 and all codewords of $C^{\perp}$, it is easy to check that the results are correct.

According to Definition 4.2, we have $E_C(Y)=1+y_1^2y_2^2$, and then by Corollary 4.5,

$$E_{C^{\perp}}(Y) = \frac{1}{|C|}\sum_{p(x)\in C}\prod_{k=1}^{1}\prod_{j=1}^{2}[1+\sum_{i=1}^{2-1}\chi(i2^{k-1}p_{2-j}) y_j^{i2^{k-1}}][1+\sum_{i=1}^{2-1}\chi(i2^{k-1}p_{2-j}v) y_j^{i2^{1+k-1}}]$$

$$= \frac{1}{2}[(1+y_1+y_1^2+y_1^3)(1+y_2+y_2^2+y_2^3)+(1-y_1-y_1^2+y_1^3)(1-y_2-y_2^2+y_2^3)]$$

$$= 1+y_1y_2+y_1^3+y_1^2y_2+y_1y_2^2+y_2^3+y_1^2y_2^2+y_1^3y_2^3.$$

On the other hand, according to Definition 4.2 and list all codewords of $C^{\perp}$, it is easy to check that the results are correct.

VI. Conclusion

This paper is devoted to the determination of the MacWilliams identities of linear codes with respect to Lee complete ρ weight enumerators and exact complete ρ weight enumerators over $M_{n\times s}(Z_{l^m}+vZ_{l^m})(v^2=v)$. When $|C^{\perp}|$ is sufficiently large, we don't have to work out specific code-words of a code, according to Corollary 3.9 and Corollarys 4.5, we can obtain Lee complete ρ weight enumerator and exact complete ρ weight enumerator of a linear code over R, for a linear code over $M_{n\times s}(R)$, it is enough to consider Theorems 3.8 and 4.4.

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