Iterative Algorithm for High Resolution Frequency Estimation

Isabel M. P. Duarte, José M. N. Vieira, Paulo J. S. G. Ferreira, and Daniel F. Albuquerque

***Abstract*—Compressed sensing (CS) is a theory that allows us to recover sparse or compressible signals from a much smaller number of samples or measurements than with traditional methods. The problem of detection and estimation of the frequency of a signal is more difficult when the frequencies of the signal are not present on the DFT basis. The Fourier coefficients are not exactly sparse due to the leakage effect if the frequency is not a multiple of the fundamental frequency. In this work we present a high frequency resolution spectrum estimation algorithm that explores the CS, for this type of nonperiodic signal from finite number of samples. It takes advantage of the sparsity of the signal in the frequency domain. The algorithm transforms the DFT basis into a frame with a large number of vectors by inserting columns between some of the existing ones. The proposed algorithm can estimate the amplitudes and frequencies even when the frequencies are too close together, a particularly difficult situation which are not covered by most of the known algorithms. Simulation results show good convergence and a high resolution when compared with other algorithms.**

***Index Terms*—Compressed sensing, redundant frames, signal reconstruction, sparse representations, spectral estimation.**

I. Introduction

The compressed sensing (CS) theory allow us to recover a sparse signal x from a number of measurements M, much smaller than the length N of the signal. In CS, instead of using the Nyquist sampling rate, which leads to periodic samples, the N-length signal is acquired with a smaller rate, obtaining the measurements by means of inner products with M random vectors, $M < N$. In other words, we want to estimate the signal x, from a small number M of measurements, assuming that the signal is sparse or is sparse in a basis. The problem can therefore be reduced to seeking the sparsest solution to $y = \Phi x$, where y is the measurement vector and Φ is a $M \times N$ measurement matrix whose rows are the M random vectors.

The CS can be described by: consider $x \in \Re^N$ a signal with only K nonzero components, we want to recover it from just M known measurements, with $M > K$ and $M \ll N$.

These samples are obtained projecting x on a set of M vectors $\{\phi_i\} \in \Re^N$, that are independent of the signal, with which we can build the matrix $\Phi \in \Re^{M \times N}$. The measurements vector is given by $y = \Phi\, x$.

To reconstruct the K-sparse signal x, we must solve the problem,

$$(P0): \min \|x\|_0 : y = \Phi x, \qquad (1)$$

where $\|x\|_0$ is the number of nonzero entries of the signal x.

Since the matrix Φ is rank deficient, one can think the problem is impossible, but it can be shown that if the matrix obeys the Restricted Isometry Property (RIP), we can recover x using the **Basis Pursuit** (BP) principle. Instead of (P0) one solves its convex relaxation, [1]:

$$(P1): \min \|x\|_1 : y = \Phi x, \qquad (2)$$

where $\|x\|_1 = \Sigma\, |x_i|$.

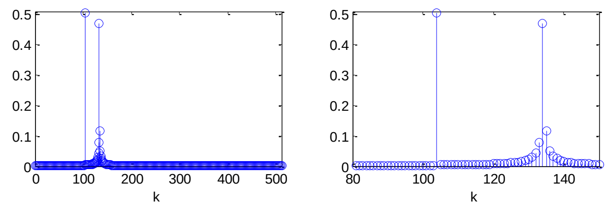
Essentially, this property requires that every set of less than K columns, behaves like an orthonormal system. More precisely, let Φ_T, $T \subset \{1, \cdots, N\}$, be the $M \times |T|$ submatrix consisting of the columns indexed by T. The K-restricted isometry constant δ_K of Ψ is the smallest quantity such that for all the subset $T \subset N$, with $T \le K$ and coefficient sequences $(x_j)_{j \in T}$. The signal x, K-sparse or compressible, can be recovered by solving the indeterminate system $y = \Phi x$, either by (P0) or (P1) from only $M = O(K \log(N/K))$ measurements, particularly with matrices with Gaussian entries, [2], [3].

$$(1-\delta_K)\|x\|^2 \le \|\Phi_T x\|^2 \le (1+\delta_K)\|x\|^2 \qquad (3)$$

If a signal x can be written as a linear combination of K sinusoids, the signal presents few non-zero spectral lines in the classical Fourier Transform sense, that is, it is K-sparse in the frequency domain.

However, in practical applications, because we use finite N-length signals, the signal is sparse only if the frequencies are multiples of the fundamental frequency $2\pi/N$, see Fig. 1.

The success of the traditional CS algorithms is compromised since there is leakage.



(a) (b)

Fig. 1. Spectrum of a signal with length N=1024 composed by two sinusoids, the first is multiple of the fundamental frequency of the DFT and the second is not. Plot on the (b) is a zoomed-in version of Fig. (a).

Here, we propose an iterative algorithm that finds a first approximate location of a sinusoid and then refine the

Manuscript received April 30, 2014; revised June 15, 2014. This work was supported by The Portuguese Foundation for Science and Technology (FCT) through the project PEst-OE/CED/UI4016/2014, and the Center for Studies in Education, Technologies and Health (CI&DETS).

Isabel M. P. Duarte and Daniel F. Albuquerque are with the School of Technology and Management of Viseu, Polytechnic Institute of Viseu, CI&DETS, Portugal (e-mail: isabelduarte@estv.ipv.pt, dfa@estv.ipv.pt).

José M. N. Vieira and Paulo J. S. G. Ferreira are with Signal Processing Lab., IEETA/DETI, University of Aveiro, Portugal.

DOI: 10.7763/IJIEE.2014.V4.475

sampling in frequency around this sinusoid until a desired accuracy. If several sinusoids have to be found, the procedure iterates as many times as the number of sinusoids.

II. Compressive Sensing Applied to Spectral Analysis

Consider a signal x composed by K sinusoids, where the frequencies are not multiples of the fundamental frequency $2\pi/N$. The DFT coefficients of the signal are not sparse. There are no columns in the matrix Ψ corresponding to the K frequencies of the signal.

If we apply the CS, the recovered signal s obtained by solving the problem [4], [5]

$$(P1):\min\|s\|_1 : y=\Phi s=\Phi\Psi s,$$

where Ψ is the transposed matrix of the DFT, will not be sparse and the error is large even if we increase the number of measurements.

The natural idea to overcome this problem is to extend the matrix Ψ, obtaining a redundant frame instead of the DFT basis, such that each signal frequency is represented by a column of the matrix. Thereby the signal x will be sparse in this frame.

Now a question arises: how many columns to add and where?

Concerning the number of columns to add, the ideal would be to add a number of columns equal to the sparsity K, or just a little more, because this value is relatively small.

By setting Ψ_R the frame obtained from adding some columns to the matrix Ψ, it can be shown that the system of equations $x = \Psi_R s$ is well conditioned.

About the columns to add, if we knew, among which frequencies multiple of the fundamental frequency of the DFT is each frequency of the signal, that is the same of saying between which columns of the matrix Ψ are the frequencies of the signal, we could just add columns only in these intervals. The problem is that, in practice, we only know the measurement signal y [6].

It is known that if the measurement matrix Φ satisfies the RIP, $\Phi^T\Phi \approx I$ and then the vector $z = \Phi^T\Phi x$ is a good estimate for a K-sparse signal x [7]. In our case, we have $y = \Phi\Psi s$, so we can get an approximation for the argmax of s calculating the argmax $(\Phi\Psi)^T y = \Psi^T\Phi^T y$. Once found the position of the frequency with maximum absolute value, we can consider an interval centered on that value.

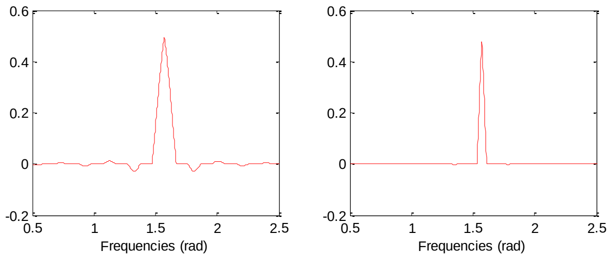
Fuchs and Deylon, in [8], presented an analytical expression of the minimal ℓ_1-norm interpolation function.

Our problem of the fractional frequency spectral estimate is dual of the fractional time-delay estimation.

This reconstruction function is very localized and as the oversampling factor, l, increases more localized it will be, as one can see in Fig. 2.

Therefore, we propose a reconstruction algorithm that takes advantage of the compactness of the interpolation function that results from the ℓ_1-norm minimization of the BP.

If we have a signal with a frequency f_i, which is not multiple of the DFT fundamental frequency, the signal is not sparse, therefore there are no maximums. However, the interpolating function will have a maximum in the position of the frequency, regardless of the amount of l which is considered. If the signal has two frequencies that are not multiples of the fundamental frequency, the interpolating function has two maxima, both between the values of the frequencies with higher values obtained by BP. See the example depicted in Fig. 3.



(a) Oversampling factor $l = 2$ (b) Oversampling factor $l = 5$

Fig. 2. Minimal ℓ_1-norm interpolation function for $l = 2$ and $l = 5$.

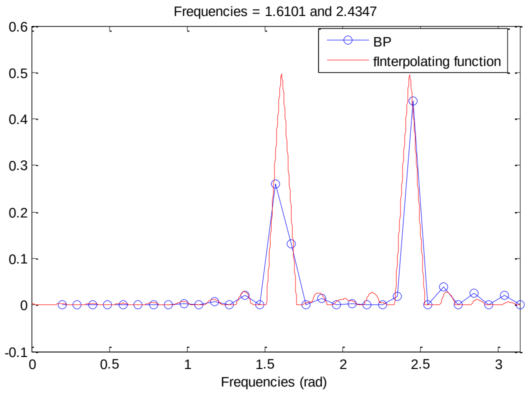


Fig. 3. Minimal ℓ_1-norm using BP and using the ℓ_1 interpolating function.

Thus, one solution to our problem of finding the approximate value of one frequency is, after finding an interval where we know the frequency is, expand the matrix Ψ by adding columns among those corresponding to the endpoints of the interval with the aim of finding the maximizer for the BP. That value is an estimate for the position of the desired frequency.

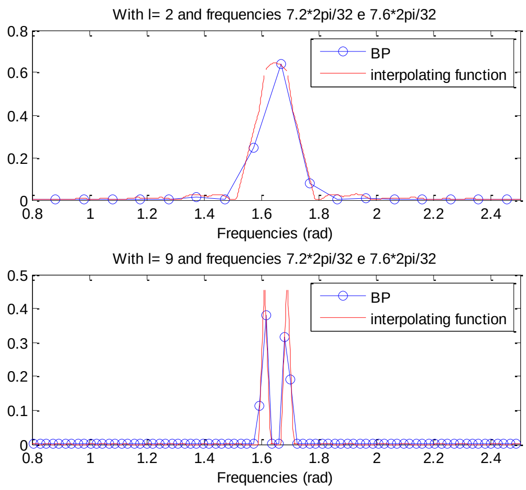


Fig. 4. Minimal ℓ_1-norm using BP and using the interpolating function with frequencies very close, using two values of l.

If the frequencies of the signal are very close, a greater value of l, or in this particular problem, a greater number of columns, must be used in order to discover them. See Fig. 4.

To determine the approximate value of another frequency, after expanding the original matrix by adding the column correspondent to the previous estimate, we find the new interval by removing the frequency found in the first approximation, that is, with $R = y - \Phi\hat{x}$.

Then, by applying again the BP we find the approximated frequency and we repeat the same procedure for all other frequencies.

The algorithm is presented in the next section.

III. Proposed Algorithm

In this algorithm we assume to know the number of the frequencies of the signal and therefore its sparsity.

We begin by choosing the redundancy factor of the frequencies of the DFT frame, c. We set the initial value of the residue: $R^{(0)} = y$.

For $k = 1$ to K do

1) Calculate the argmax of $\Psi_c^T \Phi^T R^{(k-1)}$, $f_{\max}$;
2) Consider the interval $[a, b] = [f_{\max} - \alpha,\ f_{\max} + \alpha]$;
3) Add columns between a and b:

   $I := 0$

   while (I < Nmaxpoint and ϵ > error threshold)

   - $I := I + 1$;
   - Consider the matrix Ψ_1, adding I equally spaced columns to Ψ.
   - Calculate the $\hat{s}$ values, in the interval $[a, b]$, solving the problem

     $(\text{P1}): \min \|s\|_1 : y = \Psi_1 \Phi s = \Theta s;$
   - Calculate the argmax, *freq* of $\hat{s}$, only in the interval $[a, b]$, which contain the I added columns;
   - Consider a new matrix, Ψ_2, from Ψ, where in the interval $[a, b]$ is added the column which corresponds to the argmax of the values obtained in the previous point;
   - Calculate the $\hat{s}$ values solving the problem (P1) with the matrix Ψ_2;
   - Compute the value of ϵ;
4) $\Psi = \Psi_2$;
5) $\hat{x} = \Psi\hat{s}$;
6) Obtain an estimate for the amplitude, A, of the found frequency *freq*;
7) Consider the signal x_k with one sinusoid with amplitude A and frequency *freq*;
8) $R^{(k)} = R^{(k-1)} - \Phi\, x_k$;

The approximated value of x, $\hat{x}$ is the obtained in 5.

In this algorithm, we use the standard error, given by $\|x - \hat{x}\|/\|x\|$. The parameter α was set empirically to 0.3. An excessive increase in α will reduce the algorithm efficiency and accuracy due to a wide search range; whereas a reduction of α could not include the frequency in the search range, resulting into a wrong frequency estimation. The stopping criterion in the reconstruction of the approximate value for each frequency, ϵ, i.e, the criteria used to stop adding columns in the range $[a, b]$ is given by the difference between the errors obtained in two consecutive iterations. In each iteration the error is given by the sum of absolute values of $\hat{s}$ excluding the K higher values, with K the value of sparsity. If in the interval $[a, b]$, on the step 3f, we add the column corresponding to the frequency of the sinusoid, this error is very small [6].

IV. Results

To study the performance of an algorithm we should analyze the reconstruction error of a signal and the error of the estimated values of the frequencies. Nevertheless, in this work we only present the performance of the frequency values estimation. To analyze the accuracy of that estimation, we have defined a way to measure the error of the obtained values: $e_f = d_1 + d_2 + \cdots$, where d_1 is the smallest distance between the exact frequencies and the estimated frequencies, is the smallest distance between the exact frequencies and the estimated frequencies not considered before and so on.

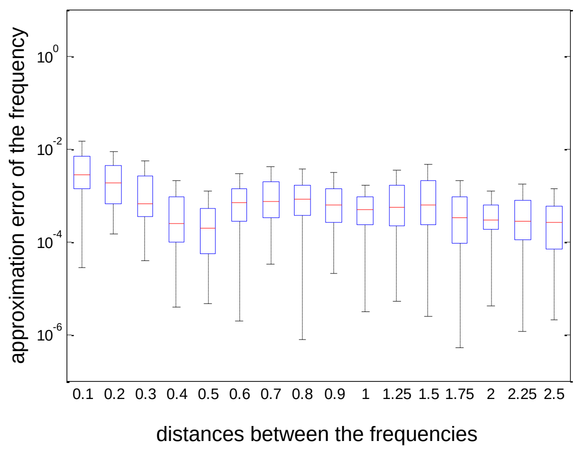
Moreover, the results of the performance for the proposed algorithm will be compared with the performance of two known algorithms proposed by *M.* Duarte *et al.* in [9], which they called Spectral CS (SCS): (1) the Spectral Iterative Hard Thresholding (SIHT) using a heuristic approximation and (2) the Root Music.

A. Performance in Estimating the Values of the Frequencies

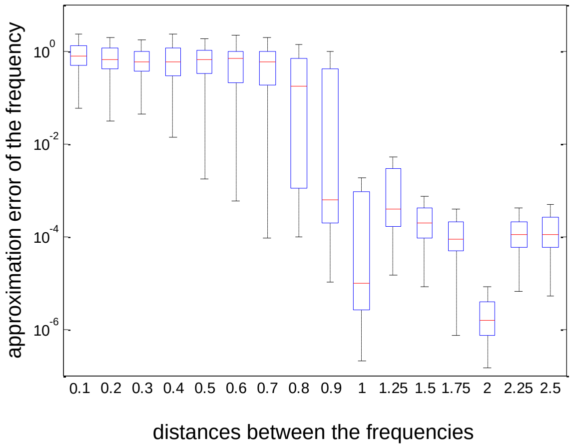
We will show that the proposed algorithm presents a better performance in estimating the frequency values than the algorithms SIHT and Root Music.

Firstly, we are going to compare the error in the case where the frequencies are close. For that propose, we consider a signal composed by two sinusoids, being the frequency of one of them f_1, fixed, and the other $f_2 = f_1 + \delta$, with δ = [0.1:0.1:1, 1.25:0.25:2.5], both with amplitude equal to one.

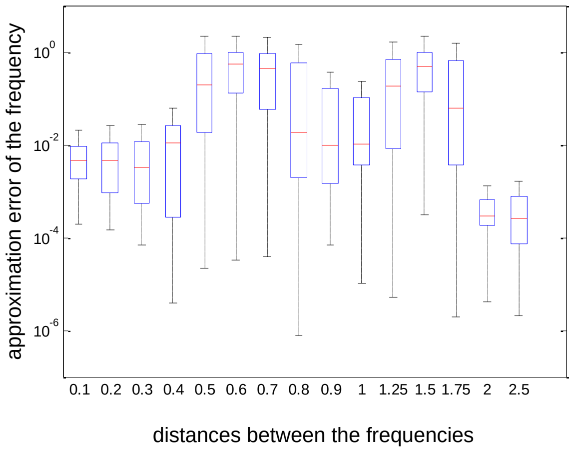
The box plots presented in (b) and (c) of Fig. 5 show that for small values of δ both the Root Music and the SIHT find estimations for the frequency values quite different from the exact values. For example, it was obtained for the frequencies f_1 = 463.7655 and $f_2 = f_1 + 0.2$ = 463.9655, the values 285.4111 and 463.8631 using the Root Music and 0 and 463.8652 using the SIHT, which are very different from the exact ones.



(a) Proposed algorithm



(b) Root music



(c) SIHT

Fig. 5. Frequency estimation error for different distance between frequencies. All quantities are averaged over 200 independent trials and it was considered M = 150 measurements.

The frequency errors were calculated only for the cases where two frequencies different to zero were found. The SIHT was the only algorithm where that did not always happened.

Note that when the two exact frequencies are close, the algorithms consider in the reconstruction, as if there was only one frequency and estimate the amplitude of one of the frequencies nearly to zero. In our example, for the Root Music, the amplitudes are 0.0249 and 0.9481 respectively and for the SIHT, 0.0023 and 0.9494 respectively.

B. Performance in Estimating the Amplitude of Frequencies

Additionally, we have also compared the performance in the case where the signals were composed by sinusoids with very different amplitudes. For that propose we have considered a signal composed by two sinusoids with amplitudes of $A_1 = 0.1$ and $A_2 = 0.01$ respectively and random frequencies. Fig. 6 presents the box plot for amplitude estimations obtained in the reconstruction of each signal.

As one can observe the worst performance in the signal reconstruction was obtained with SIHT algorithm. The Root Music presents good results for both amplitudes, however the proposed algorithm presents similar results for the amplitude 0.1 and excellent results for the amplitude 0.01.

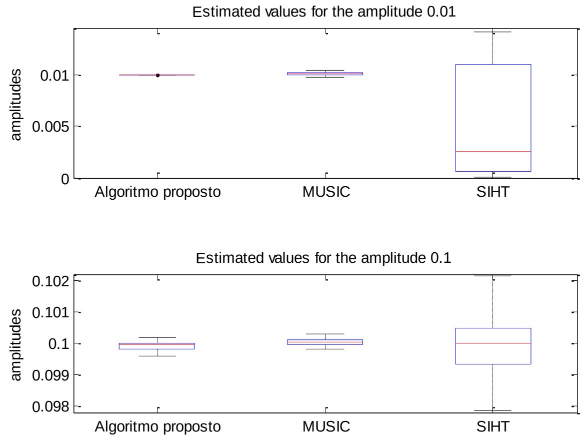


Fig. 6. Values obtained for the amplitudes in the reconstruction of a signal composed by two sinusoids using the tree algorithm with amplitude 0.1 and 0.01 considering M = 150 measurements.

V. Conclusion

The proposed algorithm can estimate the values of the frequencies with good accuracy for signals composed by several sinusoids with very close frequencies, as well as for signals composed by several sinusoids with different amplitudes.

The algorithm is particularly useful to estimate the frequencies when they are very close to each other especially in the case where the frequency differences are less than $2\pi/N$, with N the size of the signal, a particularly difficult situation for which the other known algorithms fail.

The SIHT was the less accurate and presented the worst error in the signal reconstruction. The Root Music is able to find the frequencies if they are spaced more than 1, but showed a higher signal reconstruction error. This is because the Root Music does not find the correct amplitudes. The algorithm proposed in this paper outperforms the Root Music since it works well even if the frequencies are close to each other, and gives a smaller reconstruction error.

Additionally, the proposed algorithm also presents very accurate results in the frequency estimations for signals composed by sinusoids with very different amplitudes. In all the situations described, the reconstruction error can be considered very good.

References

[1] D. L. Donoho and X. Huo, "Uncertainty principles and ideal atomic decomposition," *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 2845 – 2862, 2001.

[2] E. J. Candès and M. B. Wakin, "An introduction to compressive sampling," *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 21–30, 2008.

[3] J. A. Tropp and A. C. Gilbert, "Signal recovery from random measurements via orthogonal matching pursuit," *IEEE Transactions on Information Theory*, vol. 53, no. 12, pp. 4655–4666, Dec. 2007.

[4] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[5] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, 2006.

[6] I. Duarte, J. M. N. Vieira, P. J. S. G. Ferreira, and D. F. Albuquerque, "High resolution spectral estimation using bp via compressive sensing," in *Proc. the World Congress on Engineering and Computer Science*, 2012, vol. 1, pp. 699–704.

[7] D. Needell and J. A. Tropp, “Cosamp: Iterative signal recovery from incomplete and inaccurate samples,” *Applied and Computational Harmonic Analysis*, vol. 26, no. 3, pp. 301–321, 2009.
[8] J. J. Fuchs and B. Delyon, “Minimal L1-norm reconstruction function for oversampled signals: applications to time-delay estimation,” *IEEE Transactions on Information Theory*, vol. 46, no. 4, pp. 1666–1673, 2000, Journal.
[9] M. F. Duarte and R. G. Baraniuk, “Spectral compressive sensing,” *Applied and Computational Harmonic Analysis*, vol. 35, no. 1, pp. 111–129, July 2013.

Isabel M. P. Duarte was born in Viseu, Portugal. She received her diploma in mathematics in 1987 from the University of Coimbra and in 2013 she received her PhD degree in electrical engineering from the University of Aveiro. She is a professor in the School of Technology and Management of Viseu (Polytechnic Institute of Viseu).

She is a member of Center for Studies in Education, Technologies and Health (CI&DETS). Her research interests include signal reconstruction, compressive sensing.

José M. N. Vieira was born in Aveiro, Portugal in 1963 and received the diploma in electrical engineering in 1988 from the University of Coimbra. In 2000 he received his PhD degree in electrical engineering from the University of Aveiro. He has been a professor of electrical engineering in the University of Aveiro since 1991, and a researcher at the IEETA Institute.

He has been the president of the Portuguese section of the AES, since 2003. His major research interests are in the fields of digital audio, signal reconstruction, digital fountains and compressed sensing.

Paulo J. S. G. Ferreira was born in Torres Novas, Portugal, and is a professor Catedrático at the Departamento de Electrónica, Telecomunicações e Informática, Universidade de Aveiro. He was an associate editor of the IEEE Transactions on Signal Processing, and he is currently a Member of the Editorial Board of the Journal of Applied Functional Analysis and Editor-in-Chief of Sampling Theory in Signal and Image Processing.

He co-edited (with J.J. Benedetto) Modern Sampling Theory: Mathematics and Applications. His research interests include sampling, coding, signal reconstruction, and bioinformatics.

Daniel F. Albuquerque was born in Aveiro, Portugal in 1984. He received his diploma in electronics and telecommunications engineering (2007) and PhD in electrical engineering (2013) from University of Aveiro (Portugal). He was an invited assistant in the School of Technology and Management of Viseu (Polytechnic Institute of Viseu) from 2011 to 2013 and an invited professor in the school since 2013.

He is a member of Center for Studies in Education, Technologies and Health (CI&DETS). He is a member of IEEE. He receives the 1st Plug Prize from APRITEL in 2010 and 3rd Prize in Technological Realization in Audio Engineering from the Portuguese section of Audio Engineering Society in 2007.