

Finite-Time Control of Unified Chaotic Systems via I&I

Qiyue Xie, Xiaoli Wang, and Yi Zuo

Abstract—This paper deals with the finite-time chaos control problem of unified chaotic systems. Based on the finite-time stability theory and system immersion and manifold invariance (I&I) which is a geometric order reducing control approach, a nonlinear control law is proposed to stabilize the unified chaotic system in a finite time. The control law is designed by immersing the unified chaotic system into a one-dimensional finite-time stable system. In the process of control design, this approach does not require the knowledge of Lyapunov function. And it is able to deal with finite-time control problem of nonlinear system with high dimension. The general results obtained in the paper are proved through theory analysis. Simulation results are provided to verify the effectiveness of the presented scheme.

Index Terms—Unified chaotic systems, finite-time control, I&I.

I. INTRODUCTION

Chaotic system has attracted increasing attention since it has potential application in various fields such as mechanical engineering, secure communication, electrical engineering, information processing, biological engineering and chemical processing [1]-[3]. Stimulated by the work of Ott, Grebogi and Yorke (OGY) [4], chaos control has become one of research focuses in recent years [5]-[9]. Many methods (such as backstepping method [5], adaptive control [6], sliding mode control [7], control Lyapunov function method [8], etc.) have been proposed to stabilize the chaotic system asymptotically. And nonsingular terminal sliding mode control is also adopted to study finite-time stabilization of the chaotic system [9]. Most of these methods rely on the Lyapunov function, and are unable to deal with high order chaotic system.

System immersion and manifold invariance (I&I), a newly proposed reduced-order geometry method, which is able to deal with the stabilization problem of nonlinear high dimension system, does not rely on the Lyapunov function [10]. Therefore, in order to overcome the above difficulties, this paper introduces the I&I method to deal with the unified chaotic systems which is a generalized form of the L \ddot{u} system, Chen system, and Lorenz system [9]. Based on the finite-time stability theory and I&I approach, we propose a new controller to realize finite-time chaos control for the unified chaotic systems.

Notations: \mathcal{R}^n represents the n -dimensional real Euclidean

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Space.

II. PRELIMINARY DEFINITIONS AND LEMMA

Definition 1. Consider the following dynamic system:

$$\dot{x}(t) = f(x(t)), \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the system state. If there exists a constant $T > 0$ (T may depend on the initial state $x(0)$), such that $\lim_{t \rightarrow T} x(t) = x_*$, and $x(t) \equiv x_*$, if $t \geq T$, then the system (1) is finite-time stable(system(1) has a finite-time stable equilibrium point $x = x_*$)[11].

Lemma 1. Consider a system described by

$$\dot{x} = f(x, u), \quad (2)$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, with an equilibrium point $x_* \in \mathcal{R}^n$ to be stabilized, and $f(\cdot, \cdot) : \mathcal{R}^n \times \mathcal{R}^m \rightarrow \mathcal{R}^n$. If there are an integer p such that $0 < p < n$ and five mappings: $\alpha(\cdot) : \mathcal{R}^p \rightarrow \mathcal{R}^p$, $\pi(\cdot) : \mathcal{R}^p \rightarrow \mathcal{R}^n$, $c(\cdot) : \mathcal{R}^p \rightarrow \mathcal{R}^m$, $\phi(\cdot) : \mathcal{R}^n \rightarrow \mathcal{R}^{n-p}$, and $\psi(\cdot, \cdot) : \mathcal{R}^n \times \mathcal{R}^{n-p} \rightarrow \mathcal{R}^m$ such that the following hold.

(A1) The system

$$\dot{\xi} = \alpha(\xi), \quad (3)$$

where $\xi \in \mathcal{R}^p$ has a finite-stable equilibrium point $\xi_* \in \mathcal{R}^p$.

(A2) π is an injection satisfying $x_* = \pi(\xi_*)$. And for all $\xi \in \mathcal{R}^p$

$$f(\pi(\xi), c(\pi(\xi))) = \frac{\partial \pi}{\partial \xi} \alpha(\xi). \quad (4)$$

(A3) ϕ is a surjection. It makes the set identity

$$M = \{x \in \mathcal{R}^n \mid \phi(x) = 0\} = \{x \in \mathcal{R}^n \mid x = \pi(\xi), \xi \in \mathcal{R}^p\} \quad (5)$$

hold.

(A4) For a bounded initial conditions set, all trajectories of the system

$$\dot{z} = \frac{\partial \phi}{\partial x} f(x, \psi(x, z)), \quad (6)$$

$$\dot{x} = f(x, \psi(x, z)), \quad (7)$$

are uniformly bounded and system (6) has a finite-stable equilibrium point $z=0$.

Then x_* is a finite stable equilibrium point of the following closed-loop system

$$\dot{x} = f(x, \psi(x, \phi(x))).$$

Proof: From conditions (A1), (A2) and (A3), it is easy to

draw the conclusion that M is an invariant manifold. Now, we establish the claim in two steps.

When state x is off the manifold M , by the condition (A4), there exist constant T_1 such that $\lim_{t \rightarrow T_1} z(t) = 0$, and $z(t) \equiv 0$, if $t \geq T_1$.

Namely, state x will get into the manifold in a finite time T_1 .

Once state x is on manifold, it is necessary to check whether state x converges to the equilibrium point x_* in a finite time or not. By (A1), there exists a constant T_2 such that

$$\lim_{t \rightarrow T_2} \xi(t) = \xi_*, \text{ and } \xi(t) \equiv \xi_*, \text{ if } t \geq T_2.$$

According to conditions (A1), (A2) and (A3), $x = \pi(\xi)$ on the manifold and $x_* = \pi(\xi_*)$, then

$$x_* = \pi(\xi_*) = \lim_{t \rightarrow T_2} \pi(\xi(t)) = \lim_{t \rightarrow T_2} x(t).$$

So $\lim_{t \rightarrow T_2} x(t) = x_*$, and $x(t) \equiv x_*$, if $t \geq T_2$.

Therefore, there exist a constant $T = T_1 + T_2$ such that

$$\lim_{t \rightarrow T} x(t) = x_*, \text{ and } x(t) \equiv x_*, \text{ if } t \geq T.$$

Namely, x_* is a finite stable equilibrium point of the closed-loop system

$$\dot{x} = f(x, \psi(x, \phi(x))).$$

The proof is completed.

Remark 1: By Lemma 1, the problem of I&I finite-time control can be decomposed into two sub problems. First, give the finite-stable target system, and find, if possible, a manifold M described implicitly by $\{x \in \mathcal{R}^n | \phi(x) = 0\}$ and in parameterized form by $\{x \in \mathcal{R}^n | x = \pi(\xi), \xi \in \mathcal{R}^p\}$, which can be rendered invariant with internal dynamics a copy of the target system. Second, construct a control law $u = \psi(x, z)$ that drives the off-the-manifold coordinate $z = \phi(x)$ to zero in finite time and keeps the trajectory of system (7) bounded.

III. CHAOS CONTROL DESIGN FOR UNIFIED CHAOTIC SYSTEM

Consider the controlled unified chaotic system as follows [9]:

$$\begin{cases} \dot{x}_1 = (25\alpha + 10)(x_2 - x_1) + u_1, \\ \dot{x}_2 = (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2 + u_2, \\ \dot{x}_3 = x_1x_2 - (8 + \alpha)x_3 / 3 + u_3, \end{cases} \quad (8)$$

where $x_1, x_2, x_3 \in \mathcal{R}$ are the states, $u_1, u_2, u_3 \in \mathcal{R}$ are the control inputs, and parameter $\alpha \in [0, 1]$. When $u_1 = u_2 = u_3 = 0$ in the system (10), there will exist Lorenz attractor when the parameter satisfies $0 \leq \alpha \leq 0.8$, there will exist Lü attractor when the parameter satisfies $\alpha = 0.8$, there will exist Chen attractor when the parameter satisfies

$$0.8 < \alpha \leq 1.$$

Using Lemma 1, we choose a finite-time stable system $\dot{\xi} = -\theta\xi^\eta$ ($\xi \in \mathcal{R}$, $\theta > 0$, $\eta = p/q$, and $p < q$ are positive odd integers) as the target system. Condition (A1) holds according to the results on Reference[9].

Define $\pi_1 = \xi$, $\pi_2 = -\xi$, $\pi_3 = \xi$, then, equation (3) of (A2) becomes

$$\begin{cases} \theta\xi^\eta = -2(25\alpha + 10)\xi + c_1, \\ -\theta\xi^\eta = (28 - 35\alpha)\xi - \xi^2 - (29\alpha - 1)\xi + c_2, \\ \theta\xi^\eta = -\xi^2 - (8 + \alpha)\xi / 3 + c_3. \end{cases}$$

From these equations, we can obtain

$$\begin{cases} c_1 = \theta\xi^\eta + 2(25\alpha + 10)\xi, \\ c_2 = -\theta\xi^\eta - (28 - 35\alpha)\xi + \xi^2 + (29\alpha - 1)\xi, \\ c_3 = \theta\xi^\eta + \xi^2 + (8 + \alpha)\xi / 3. \end{cases}$$

Now, the manifold $x = \pi(\xi)$ can be implicitly described by

$$\phi(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \end{pmatrix} = 0.$$

So, conditions (A3) have been constructed according to Lemma 1.

Let $z = \phi(x)$. Differentiating z with respect to time, the off-the-manifold dynamics is given as follows

$$\begin{cases} \dot{z}_1 = (25\alpha + 10)(x_2 - x_1) + (28 - 35\alpha)x_1 - x_1x_3 \\ \quad + (29\alpha - 1)x_2 + \psi_1 + \psi_2, \\ \dot{z}_2 = (25\alpha + 10)(x_2 - x_1) - x_1x_2 + (8 + \alpha)x_3 / 3 \\ \quad + \psi_1 - \psi_3. \end{cases} \quad (9)$$

Then, we can design controller as follows

$$\begin{cases} \psi_1 = -(25\alpha + 10)(x_2 - x_1) - \theta x_1^\eta, \\ \psi_2 = -(28 - 35\alpha)x_1 + x_1x_3 - (29\alpha - 1)x_2 + \theta x_1^\eta - \theta z_1^\eta, \\ \psi_3 = -x_1x_2 + (8 + \alpha)x_3 / 3 - \theta x_1^\eta + \theta z_2^\eta. \end{cases}$$

Substituting the controller into system (9), we can obtain

$$\begin{cases} \dot{z}_1 = -\theta z_1^\eta, \\ \dot{z}_2 = -\theta z_2^\eta, \end{cases} \quad (10)$$

$$\begin{cases} \dot{x}_1 = -\theta x_1^\eta, \\ \dot{x}_2 = \theta x_1^\eta - \theta z_1^\eta, \\ \dot{x}_3 = -\theta x_1^\eta + \theta z_2^\eta. \end{cases} \quad (11)$$

It is natural that condition (A4) holds.

Therefore, the I&I finite-time controller is

$$\begin{cases} \psi_1 = -(25\alpha + 10)(x_2 - x_1) - \theta x_1^\eta, \\ \psi_2 = -(28 - 35\alpha)x_1 + x_1x_3 - (29\alpha - 1)x_2 \\ \quad + \theta x_1^\eta - \theta(x_1 + x_2)^\eta, \\ \psi_3 = -x_1x_2 + (8 + \alpha)x_3 / 3 - \theta x_1^\eta + \theta(x_1 - x_3)^\eta. \end{cases}$$

IV. NUMERICAL SIMULATIONS

To demonstrate the effectiveness of the I&I finite-time stabilization controllers proposed in this section, we present the simulation results for the unified system.

In the numerical simulation of the unified system under control based on I&I approach, it has been carried out with the following coefficient and the initial conditions

$$\alpha = 0.5, x(0) = [1 \quad 1 \quad -1]^T.$$

Simulation results without control are shown in Fig. 1 and Fig. 2. Simulation result under I&I finite-time control is shown in Fig. 3.

From Fig. 1 and Fig. 2, there exists Lorenz attractor in the unified system. Fig. 3 shows that all of the state trajectories in the unified system are droved to zero in a finite time. Therefore, the I&I finite-time approach can effectively solve the finite-time control problem of unified chaotic system.

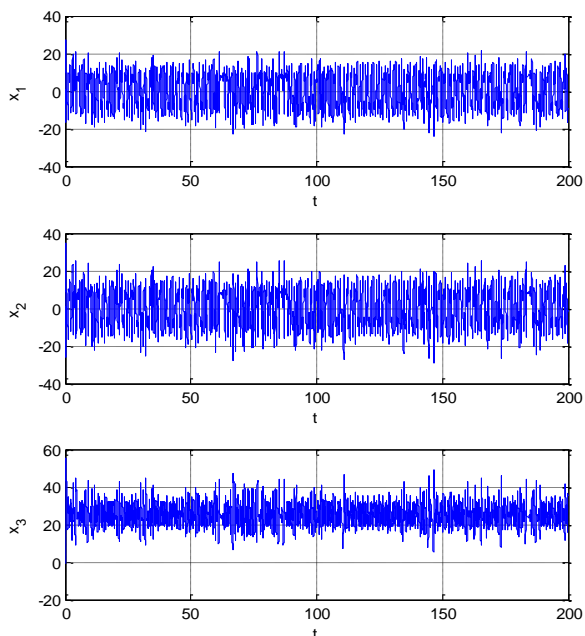


Fig. 1. State trajectories of system (10) without control.

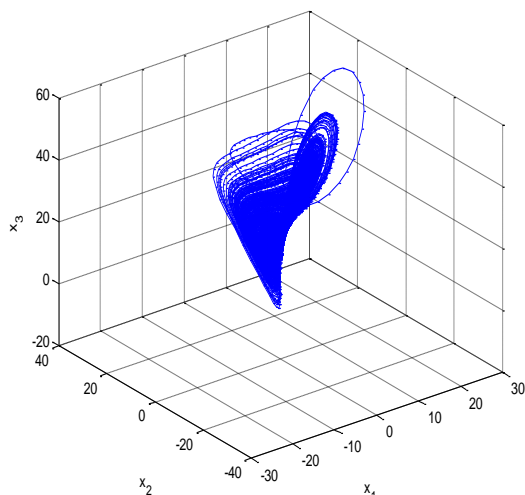


Fig. 2. Lorenz attractor in 3D.

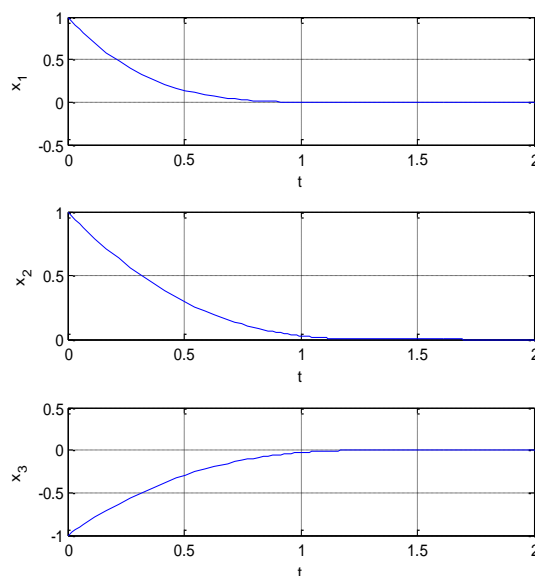


Fig. 3. State trajectories of system (10) with I&I finite-time control.

V. CONCLUSION

In this paper, I&I approach is introduced to design a feedback controller for the unified chaotic systems. The finite-time controller is designed by immersing the unified chaotic system into a one-dimensional finite-time stable system. This approach holds two advantages: first, it does not rely on the Lyapunov function; second, it is able to deal with high dimension systems through system immersion. This is illustrated by the above discussion and we are sure that this technique can be extended to other high-dimensional chaotic systems easily.

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