L-Moment and Inverse Moment Estimation of the Inverse Generalized Exponential Distribution

Muhammad Shuaib Khan

***Abstract*—This paper presents the L-Moment and Inverse Moment estimation of the inverse Generalized Exponential distribution also the properties of the L-Moment. The main interests are in the relationship between β and various L-Moment; measure of variability for L-Moment as the numerical quantities that describe the spread of the values in a set of data. Here these L-Moment models presented graphically and mathematically. The Inverse Moment estimation models are MTTF, Coefficient of Variation, Coefficient of Skewness and Coefficient of kurtosis are presented mathematically.**

***Index Terms*—Inverse Generalized Exponential distribution, l-moments, l-moments estimation, l-coefficient of variation, l-skewness, l- kurtosis, inverse moment estimation.**

I. Introduction

Recently a new two parameter distribution, named as Inverse Generalized Exponential (IGE) probability distribution has been introduced by the author M. Shuaib Khan [1], [2], [3], [4]. The Inverse Generalized Exponential probability distribution is the newly developed probability distribution has two parameters β and η. It can be used to represent the failure probability density function (PDF) and is given by

$$f_{IGE}(t)=\frac{\beta}{\eta}\left(\frac{1}{t-t_0}\right)^2 Exp\left[-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)\right]^{\beta},\ \eta>0,\beta>0,t_0>0,t_0<t<\infty \qquad (1.1)$$

and the corresponding cdf is

$$F_{IGE}(t)=Exp\left[-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)\right]^{\beta} \qquad (1.2)$$

where β is the shape parameter representing the different pattern of the Inverse Generalized Exponential PDF and is positive and η is a scale parameter representing the characteristic life at which $(36.8)^{\beta}$% of the population can be expected to have failed and is also positive, t_0 is a location or shift or threshold parameter (sometimes called a guarantee time, failure-free time or minimum life). It is important to note that the restrictions in eq. (1.1) on the values of t_0, η, β are always the same for the Inverse Generalized Exponential distribution.

2011; revised July 17, 2011.

Muhammad Shuaib Khan is with Department of Statistics, The Islamia University of Bahawalpur Pakistan, Bahawalpur, and GPO 63100 PK (e-mail: shuaib.stat@gmail.com).

Fig. 1.1 shows the diverse shape of the Inverse Generalized Exponential PDF with $t_0=0$ and value of $\eta=1$ and β (=0.8, 1, 1.5, 2, 2.5).

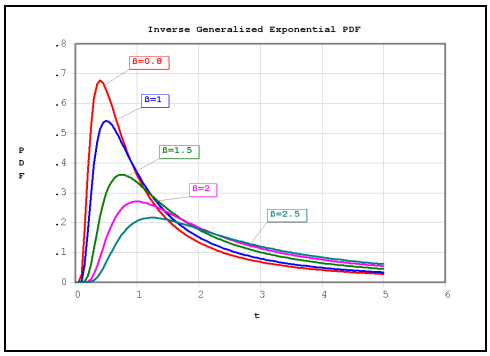


Fig 1.1 The Inv Gen Expo PDF

Fig. 1.1 shows the diverse shape of the Inverse Generalized Exponential PDF which is quite similar with the shape of Inverse weibull distribution. The Inverse Generalized Exponential and the Inverse weibull distributions are both the generalization of an inverse exponential distribution.

II. L-Moments

As discussed in previous section, the alternative measure of distribution of shapes denoted by Hosking [5][6], L-moments are expectation of certain linear combinations of order Statistics. Hosking has defined the L-moments of X to be the quantities.

$$\lambda_r = r^{-1}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}E(X_{r-k:r}),\ \text{r} = 1, 2, 3, 4 \qquad (2.1)$$

The L in "L-moments" emphasizes that λ_r is a linear function of the expected order statistics. The expectation of an order Statistics has been written as Hosking [5]

$$EX_{j:r}=\frac{r!}{(j-1)!(r-j)!}\int x(F)\{F(x)\}^{j-1}\{1-F(x)\}^{r-j}dF(x). \qquad (2.2)$$

The first few L-moments, λ_r of random variable"X", as defined by Hosking [1] are given below:

$$\lambda_1 = E(X)=\int_0^1 x(F)dF$$

$$\lambda_2=\frac{1}{2}E\left(X_{2:2}-X_{1:2}\right)=\int_0^1 x(F)(2F-1)dF$$

$$\lambda_3=\frac{1}{3}E\left(X_{3:3}-2X_{2:3}+X_{1:3}\right)=\int_0^1 x(F)\left(6F^2 6F+1\right)dF$$

$$\lambda_4 = \frac{1}{4}E\left(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}\right) = \int_0^1 x(F)\left(20F^3 - 30F^2 + 12F - 1\right)dF$$

where $X_{k:n}$ is an order Statistics, the k^{th} smallest of a sample of size "n" drawn from the distribution of X and $x(F)$ is a quantile function of real-valued random variable X. The "L" in L-moment emphasized that λ_r is a linear function of expected order Statistics. Measures of skewness and kurtosis, based on L-moments, are respectively.

$$\text{L-skewness} = \tau_3 = \lambda_3/\lambda_2 \tag{2.3}$$

And

$$\text{L-Kurtosis} = \tau_4 = \lambda_4/\lambda_2 \tag{2.4}$$

The L-moments $\lambda_1, \cdots, \lambda_r$ and L-moment ratios $\tau_3, \cdots, \tau_r$ are useful quantities for summarizing a distribution. The L-moments are in some ways analogous to the (conventional) center moments and the L-moments ratios are analogous to moment's ratios. In particular $\lambda_1, \lambda_2, \tau_3$ *and* τ_4 may be regarded as measures of location, scale, skewness and kurtosis respectively. Hosking [5] has shown that L-moments have good properties as measure of distributional shape and are useful for fitting distribution to data.

III. Calculation of L-skewness and L-kurtosis

The probability distribution can be summarized by the following four measures. The mean or L-location (λ_1), The L-scale (λ_2), The L-skewness (τ_3), The L-Kurtosis(τ_4), we now consider these measures, particularly τ_3 and τ_4 in more details. Moments are often used as summary measure of the shapes of a distribution. In this section the measure of skewness and kurtosis based on L-moments has been calculated from Inverse Generalized Exponential distribution. The first four L-moments of Inverse Generalized Exponential distribution given in expression (1.1) have been calculated from relations (2.2), as shown in fig. (3.1)

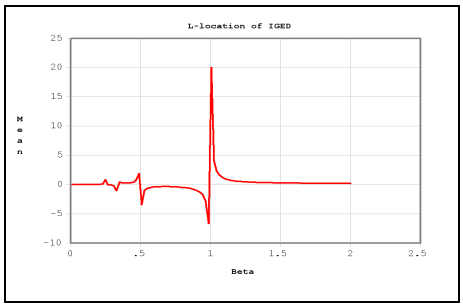


Fig 3.1 β vs L-location of IGED

$$\lambda_1 = \frac{\Gamma\left(1-\frac{1}{\beta}\right)}{\eta} \tag{3.1}$$

$$\lambda_2 = \frac{\Gamma\left(1-\frac{1}{\beta}\right)}{\eta^{\frac{1}{\beta}}}\left[2^{\frac{1}{\beta}} - 1\right] \tag{3.2}$$

$$\lambda_3 = \frac{\Gamma\left(1-\frac{1}{\beta}\right)}{\eta}\left[1 - 3(2)^{\frac{1}{\beta}} + 2(3)^{\frac{1}{\beta}}\right] \tag{3.3}$$

$$\lambda_4 = \frac{\Gamma\left(1-\frac{1}{\beta}\right)}{\eta}\left[1 + 6(2)^{\frac{1}{\beta}} - 10(3)^{\frac{1}{\beta}} + 5(4)^{\frac{1}{\beta}}\right] \tag{3.4}$$

$$\lambda_1 = \frac{\Gamma\left(1-\frac{1}{\beta}\right)}{\eta} \tag{3.1}$$

The above function describes the L-location of the inverse generalized exponential distribution. For the graphical analysis L moment of measure of central tendency is the mean denoted by λ_1. In the graphical analysis β is the shape parameter and η is a scale parameter representing the characteristic life. Fig. 2.1 shows the oscillate shape of the L-location of the Inverse Generalized Exponential with value of $\eta = 10$.

The corresponding measures of L-Coefficient of Variation, L-skewness τ_3 and L-kurtosis τ_4 using (2.3) and (2.4) are shown in fig. (3.2), (3.3), (3.4).

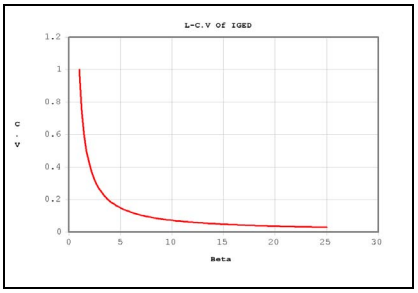


Fig 3.2 β vs $L-CV_{IGE}$

$$C.V = 2^{\frac{1}{\beta}} - 1 \tag{3.5}$$

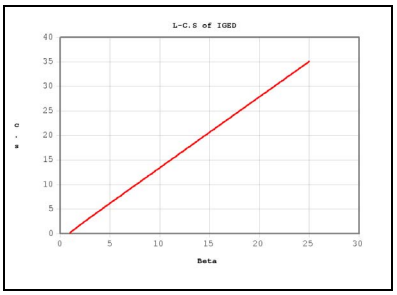


Fig 3.3 β vs $L - C.SK_{IGE}$

$$\tau_3 = \frac{1 - 3(2)^{\frac{1}{\beta}} + 2(3)^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}} - 1} \tag{3.6}$$

The L-C.V is used for checking the consistency of the data shown in Fig. 3.2. The Relationship between β and CV_{IGE} shows that as $\beta \to \infty$ then the value of CV_{IGE} decreases asymptotically. $C.SK_{IGE}$ is the quantity used to measure the

skewness of the distribution. The relationship between β and $C.SK_{IGE}$ is shown in Fig 3.3. Fig 3.3 shows that as $\beta \to \infty$ then the value of $L-C.SK_{IGE}$ increases asymptotically.

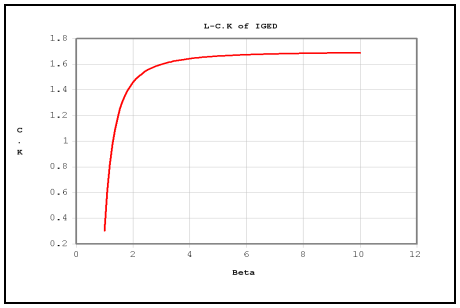


Fig 3.4 β **vs** $L-C.K_{IGE}$

$$\tau_4 = \frac{1+6(2)^{\frac{1}{\beta}} - 10(3)^{\frac{1}{\beta}} + 5(4)^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}} - 1} \qquad (3.7)$$

K_{IGE} is the quantity, which can be used to measure the kurtosis. The relationship between β and K_{IGE} is shown in Fig 3.4. Here we note that as $\beta \to \infty$ then the values of K_{IGE} is positively skewed has a maximum value asymptotically. Hence β and K_{IGE} have a positive proportion when β > 1.1.

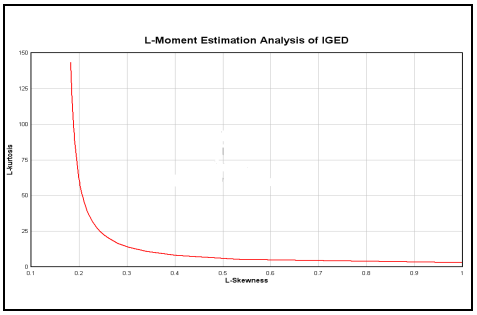


Fig 3.5 $L-C.SK_{IGE}$ vs $L-C.K_{IGE}$

The *L*-moment ratio of Inverse Generalized Exponential diagram can be used to compare the relations *between* *L*-skewness vs *L*-kurtosis. The L-moment ratio diagram shows that as skewness increases then the pattern of the kurtosis become decreases.

IV. A Numerical Illustration

TABLE I: Relationship b/w β vs L-Coefficient of Variation, L-skewness (τ3) and L-kurtosis (τ4) for Inverse Generalized Exponential Distribution

β	$L-C.V_{IGED}$	$L-C.S_{IGED}$	$L-C.K_{IGED}$
1	1	1	3
2	0.414214	0.534654	5.226225
3	0.259921	0.402953	7.979484
4	0.189207	0.341037	10.81128
5	0.148698	0.305093	13.66808
6	0.122462	0.28162	16.5358
7	0.10409	0.26509	19.40923
8	0.090508	0.25282	22.286
9	0.08006	0.243352	25.16488
10	0.071773	0.235825	28.0452
11	0.065041	0.229697	30.92651
12	0.059463	0.224612	33.80857
13	0.054766	0.220324	36.69117
14	0.050757	0.216659	39.5742
15	0.047294	0.213491	42.45757
16	0.044274	0.210726	45.3412
17	0.041616	0.20829	48.22506
18	0.039259	0.206129	51.10909
19	0.037155	0.204198	53.99328
20	0.035265	0.202463	56.87759
21	0.033558	0.200895	59.76201
22	0.032008	0.199471	62.64652
23	0.030596	0.198173	65.53111
24	0.029302	0.196983	68.41577
25	0.028114	0.19589	71.30049
26	0.027018	0.194882	74.18527
27	0.026004	0.193949	77.07009
28	0.025064	0.193083	79.95495
29	0.02419	0.192278	82.83985
30	0.023374	0.191526	85.72479
31	0.022611	0.190824	88.60976
32	0.021897	0.190166	91.49475
33	0.021227	0.189548	94.37977
34	0.020596	0.188966	97.26481
35	0.020002	0.188418	100.1499
36	0.019441	0.187901	103.0349
37	0.01891	0.187412	105.92
38	0.018408	0.186949	108.8052
39	0.017932	0.186509	111.6903
40	0.01748	0.186092	114.5754
41	0.01705	0.185696	117.4606
42	0.01664	0.185318	120.3457
43	0.01625	0.184958	123.2309
44	0.015878	0.184614	126.1161
45	0.015523	0.184286	129.0013
46	0.015183	0.183972	131.8865
47	0.014857	0.183672	134.7717
48	0.014545	0.183384	137.6569
49	0.014246	0.183108	140.5422
50	0.013959	0.182843	143.4274

Table (1) presents 50 observations for L-coefficient of variations, L-coefficient of Skewness, and L-coefficient of Kurtosis by using simulation technique.

From Table (1), one can show that

• The values of $L-C.V_{IGED}$ decreases asymptotically as $\beta \to \infty$.

• The values of $L-C.S_{IGED}$ increases asymptotically as $\beta \to \infty$.

• The values of $L-C.K_{IGED}$ increases as the value of β increases.

V. Inverse Moments Estimation of Inverse Generalized Exponential Distribution

The method of Inverse Moments Estimation of Inverse Generalized Exponential model is defined as

$$\mu'_{r^{-1}} = \frac{\eta^{r}}{\beta^{r}}\Gamma(1+r), r = 1,2,3,4,..... \quad (5.1)$$

The inverse Moments Estimator is helpful for finding the properties of the Inverse Generalized Exponential distribution. The method of inverse Moments Estimator for the parameters is

$$m'_{r^{-1}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{t^{r}{}_{i}}$$

For the convenience of display, we present the (5.1) as

$$\omega_r = \Gamma(1+r),\ r = 1,2,3,4,.....$$

$$\mu'_{r} = \frac{\eta^{r}}{\beta^{r}}\omega_{r}, r = 1,2,3,4,..... \quad (5.1a)$$

For the Rth moment estimation some of the important properties of the Inverse Generalized Exponential distribution are MTTF, Coefficient of variation, coefficient of skewness and coefficient of kurtosis.

The mean life of the Inverse Generalized Exponential distribution is

$$MTTF_{IGE} = \frac{\eta}{\beta}\omega_1 \quad (5.2)$$

The coefficient of variation CV_{IGE} life of the Inverse Generalized Exponential distribution is defined as

$$SD_{IGE} = \sqrt{\frac{\omega_2}{\omega_1^{2}} - 1} \quad (5.3)$$

From my calculation it is clear that there is unit coefficient of variation life $CV_{IGE} = 1$. The relationship between β and CV_{IGE} is fixed. The larger or the smaller the value of β has no effect on the value of CV_{IGE}.

The coefficient of skewness CS_{IGE} life of the Inverse Generalized Exponential distribution is defined as

$$CS_{IGE} = \frac{E\left(t - \frac{\beta}{\eta}\omega_1\right)^{3}}{\left(\frac{\eta^{2}}{\beta^{2}}\left(\omega_2 - \omega_1^{2}\right)\right)^{\frac{3}{2}}} \quad (5.4)$$

where $CS_{IGE} = 2$ is the quantity used to measure the skewness of the Inverse Generalized Exponential distribution, here $CS_{IGE} > 0$ so the PDF of the Inverse Generalized Exponential distribution is skewed to the right when (Mean > Median > Mode). The relationship between β and CS_{IGE} is a fix value. From my calculations it is clear that there is no CS_{IGE} life when $\beta < 0$.

The coefficient of kurtosis CK_{IGE} life of the Inverse Generalized Exponential distribution is defined as

$$CK_{IGE} = \frac{E\left(t - \frac{\beta}{\eta}\omega_1\right)^{4}}{\left(\frac{\eta^{2}}{\beta^{2}}\left(\omega_2 - \omega_1^{2}\right)\right)^{2}} \quad (5.5)$$

where $CK_{IGE} = 9$ the quantity is used to measure the kurtosis or peaked ness of the distribution. The Inverse Generalized exponential PDF shape is more peaked than the Normal PDF because the value of $CK_{IGE} > 3$.

VI. Summary and Conclusion

In this paper the quantile comprehensive study of the Inverse Generalized Exponential quantile modeling is predicted for finding the life time of the electrical and mechanical components. These patterns of β and various B-lives are helpful for finding the life of components. In this paper we also presented measure of variability for B-lives as the numerical quantities that describe the spread of the values in a set of data. Here the inverse Moments Estimator is helpful for finding the properties of the Inverse Generalized Exponential distribution.

References

[1] M. S. Khan, "Estimation of Order Statistics for Inverse Generalized Exponential Distribution," *International Transactions in Mathematical Sciences and Computer*, Vol. 2, No. 1, 2009, pp. 23-28

[2] M. S. Khan and L. Burki, "Quantile Analysis of the Inverse Generalized Exponential Distribution," International *Transaction in Applied Sciences*, Vol. 1, No. 2, 2009, pp. 57-69.

[3] M. S. Khan, "Theoretical Analysis of Inverse Generalized Exponential Models," In *Pro. of the 2009 International Conf. on Machine Learning and Computing (ICMLC 2009)*, Perth, Australia, Vol. 20, 2009, pp. 18-24.

[4] M. S. Khan, "The Beta Inverse Generalized Exponential Distribution," *International Conf. (CMIC2011) on recent development in Applied and Computational Mathematics*, Chiang Mai University, Thailand, 4-7 January, 2011.

[5] J. R. M. Hosking, "L-moments: analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal Statistical Society*, Series, B, 52, 1990, pp. 105-124.
[6] J. R. M. Hosking and J. R Wallis, Regional frequency analysis: an approach based on L-moments. *Cambridge University Press*, Cambridge, U.K, 1997.



Mr. Muhammad Shuaib Khan was born in Multan, Pakistan in 1977. He has M.Phil in Statistics, M.Sc in Statistics and (PGD) Post graduate Diploma in computer programming and computing Statistics from Bahaudin Zakaria University Multan Pakistan. He is working as a Lecturer in the Department of Statistics, The Islamia University of Bahawalpur, Pakistan. He has 35 research publications from which 22 journals publication and 13 included in conference-related papers. He has about 12 Years of research and teaching experience at undergraduate, postgraduate and research level students.

His research interests cover the broad statistical topic of Mathematical Statistics, with particular emphasis on the mathematical development and its application, distribution theory and lifetime distribution analysis, fitting distributions and goodness of fit models and engineering inference. The primary focus of his research is on lifetime distribution analysis, in particular the case where the span of life of mechanical, electrical and aeronautical components measurements variables responses.

Mr. Khan is the member of International Association of Computer Science and Information Technology (IACSIT) and member of European Network for Business and Industrial Statistics (ENBIS).